# Nonlinear Energy Harvesting

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NANOPOWER

6 partners: Wuerzburg (Ger), ICN (Sp), VTT (Fi), Univ Geneva (Ch), Unicam (It)

2.6 M€, 3 years, lead by NiPS

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N.i.P.S Laboratory

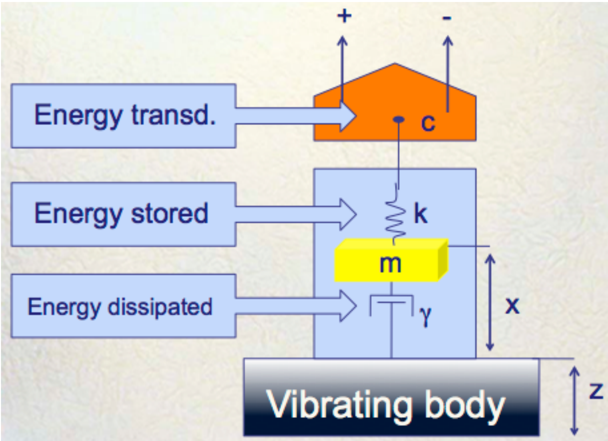
Noise in Physical Systems

ZEROPOWER

4 partners: UAB (Sp), Tyndall (Ir), Univ Glasgow (UK)

0.6 M€, 3 years, lead by NiPS

# Vibrations energy harvesting




Dynamical model

$$m\ddot{x} = -\frac{dU(x)}{dx} - \gamma\dot{x} - c(x,V) + \zeta_z$$

Where:

| | |
|---|---|
| $U(x)$ | Represents the Energy stored |
| $\gamma\dot{x}$ | Represents the dissipative force |
| $c(x,V)$ | Represents the reaction force due to the transduction mechanism |
| $\zeta_z$ | Represents the input force |

# Vibrations energy harvesting

$$\begin{cases} m\ddot{x} = -\dfrac{dU(x)}{dx} - \gamma\dot{x} - c(x,V) + \zeta_z \\ \dot{V} = F(\dot{x},V) \end{cases}$$

Details depend on the physics…

Equations that link the vibration-induced displacement with the Voltage

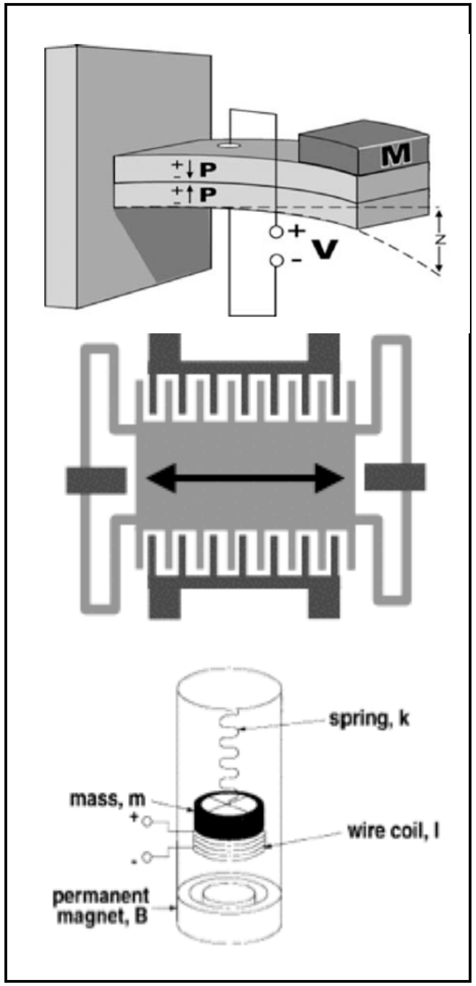
# Vibrations energy harvesting

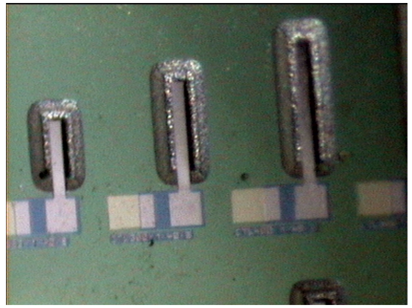
## Transduction mechanisms

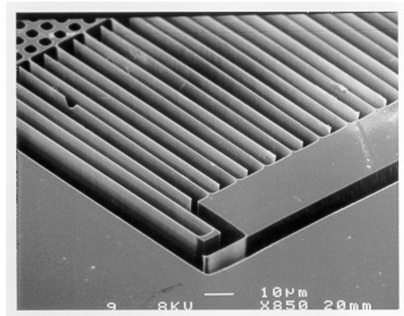
**Piezoelectric**: dynamical strain is converted into voltage difference.

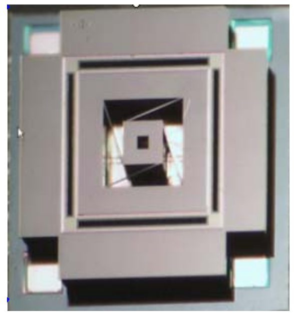
**Capacitive**: geometrical variations induce voltage difference

**Inductive**: dynamical oscillations of magnets induce electric current in coils





# Vibrations energy harvesting

Transduction mechanisms: focus on Piezo

**Piezoelectric**: dynamical strain is converted into voltage difference.

| Type | Governing Equation | Practical Maximum | Theoretical Maximum |
|---|---|---|---|
| Piezoelectric | $u = \sigma_y^2 k^2 / 2Y$ | 17.7 mJ/cm$^3$ | 335 mJ/cm$^3$ |
| Electrostatic | $u = \frac{1}{2}\varepsilon E^2$ | 4 mJ/cm$^3$ | 44 mJ/cm$^3$ |
| Electromagnetic | $u = B^2 / 2\mu_0$ | 4 mJ/cm$^3$ | 400 mJ/cm$^3$ |

# Noise energy harvesting

Transduction mechanisms

**Piezoelectric**: dynamical strain is converted into voltage difference.

$$\begin{cases} m\ddot{x} = -\dfrac{dU(x)}{dx} - \gamma\dot{x} - K_V V + \zeta_z \\ \dot{V} = K_c \dot{x} - \dfrac{1}{\tau_p} V \end{cases}$$

That for a beam are:

$$K_v = \frac{K_{eff} d_{31} a}{2 t_p k_1}$$

$$K_c = \frac{t_p d_{31} Y_p^E k_1}{a \varepsilon_p}$$

The Physics of piezo materials

Focus on $\zeta_z$

# The random character of kinetic energy

$\zeta_z$ Represents the vibration (force)

**Random vibrations** / **noise**

**Thermal noise (NOT POSSIBLE AT EQUILIBRIUM!!!)**
**Acoustic noise**
**Seismic noise**
**Ambient noise** (wind, pressure fluctuations, …)
**Man made vibrations** (human motion, machine vibrations,…)

All different for intensity, spectrum, statistics

How can we harvest them ?

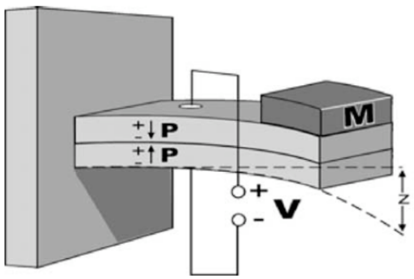
# Noise energy harvesting

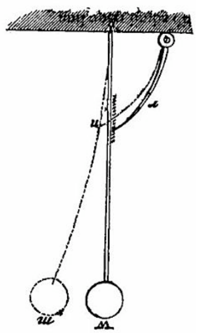
## Linear systems

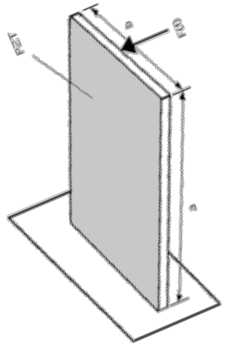
When $U(x) = \frac{1}{2}kx^2$ it is called a *linear system*

Linear systems have some interesting features… (and engineers like them most!!!)

1) There exist a simple math theory to solve the equations
2) They have a resonant behaviour (resonance frequency)
3) They can be "easily" realized with cantilevers and pendula

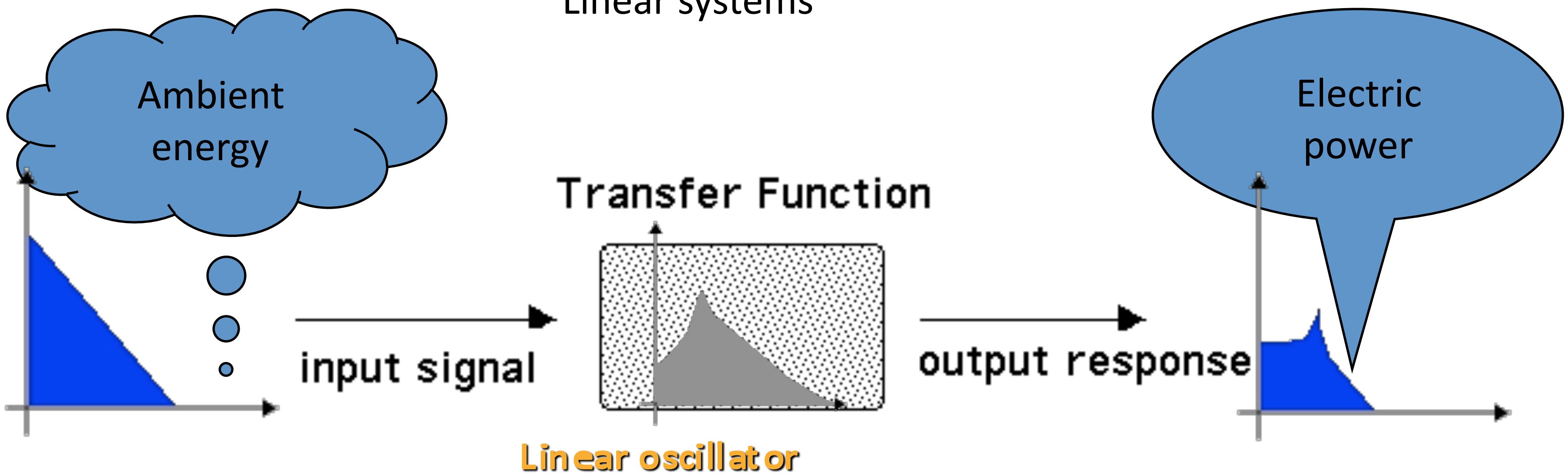




# Vibrations energy harvesting

Linear systems




The transfer function is a math function of the frequency, in the complex domain, that can be used to represent the performance of a linear system

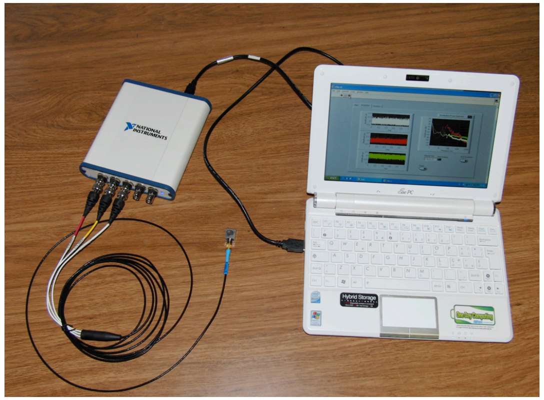
For a linear system the transfer function presents one or more peeks corresponding to the resonace frequencies and thus it is efficient mainly when the incoming energy is abundant in that regions…

This is a serious limitation when you want to build a small energy harvesting system…

# Vibrations energy harvesting

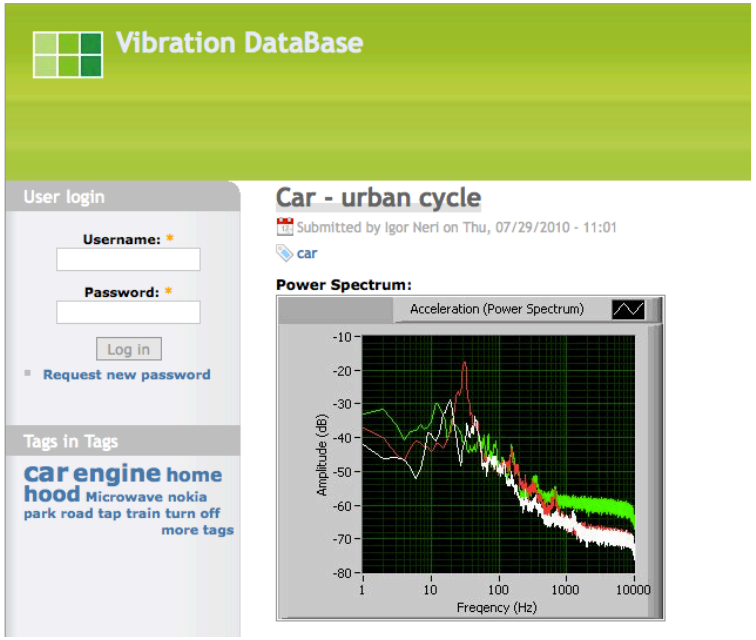
For two main reasons…

**(1)** the frequency spectrum of available vibrations instead of being sharply peaked at some frequency is usually very broad.
(See Igor Neri presentation at the workshop).



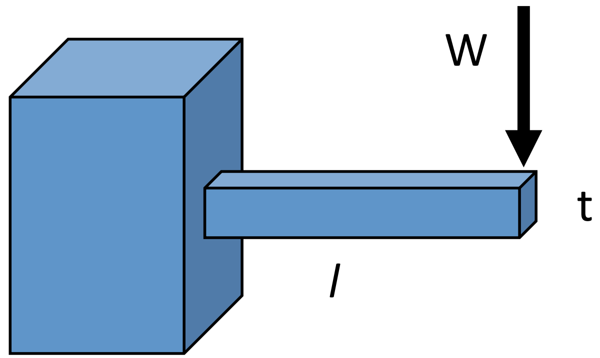
**Accelerometer:**
- Tri axial
- Bandwidth from 0.4Hz to 10kHz
- ±50g

**DAQ:**
- 102.4 kS/s five simultaneous channel
- 4 channels with software-selectable IEPE signal conditioning
- USB powered




**Signal presentation:**
- Description
- Power spectrum
- Statistical data
- Time series download (only for authorized users)

(2) The frequency spectrum of available vibrations is particularly rich in energy in the low frequency part… and it is very difficult, if not impossible, to build small low-frequency resonant systems…




Resonant frequency ~ $[\mathrm{s}^{-1}]$

- MEMS cantilever 100 × 3 × 0.1 μm³, $f_0$=12 kHz
- NEMS cantilever 0.1 × 0.01 × 0.01 μm³, $f_0$=1.2 GHz

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \delta = \frac{Wl^3}{3EI} \qquad k = \frac{W}{\delta} = \frac{3EI}{l^3}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}} = \frac{1}{2\pi}\sqrt{\frac{Ewt^3}{4Ml^3}} = \frac{t}{4\pi l^2}\sqrt{\frac{E}{\rho}}$$

Let's look an example…

## Description of the resonator design

The resonator design is a square shaped block of single crystal silicon with dimensions of 320x320x28 um^3 (design H1). Its main resonance mode is the so called square extensional (SE) resonance, which is characterized by its zoom-in/zoom-out oscillation. The resonance is excited by a piezoelectric AlN thin film on top of the resonator block. The electrically conductive (p-doped) silicon block acts as the bottom electrode, and a molybdenum thin film has been patterned to provide the top electrode. See reference [1] for a general description of the SE resonator. Reference [2] discusses piezoelectric excitation of the SE resonance mode.

Figure 3 shows how the resonator is recommended to be connected.

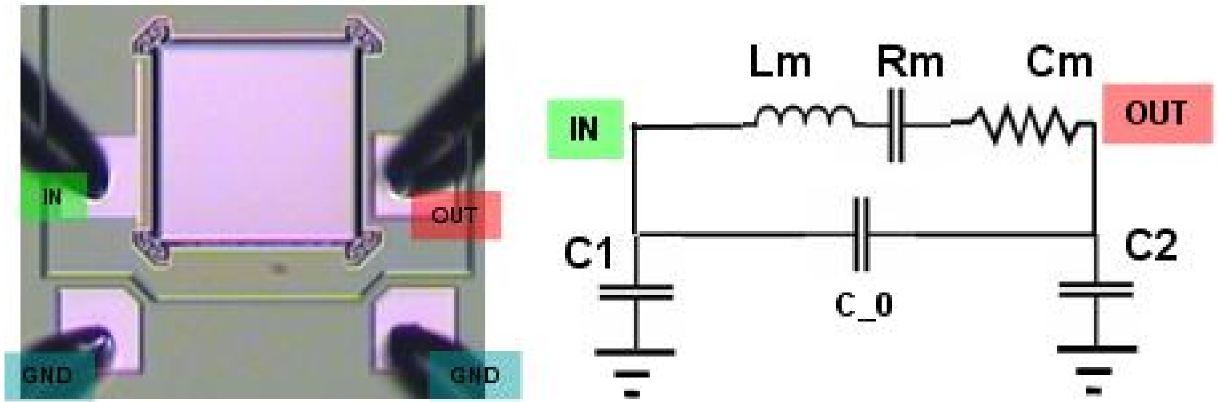
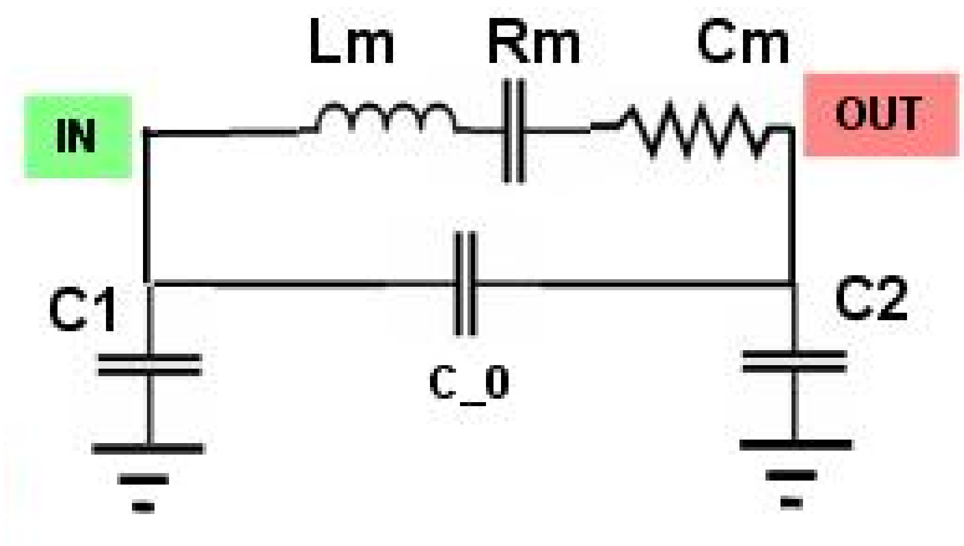


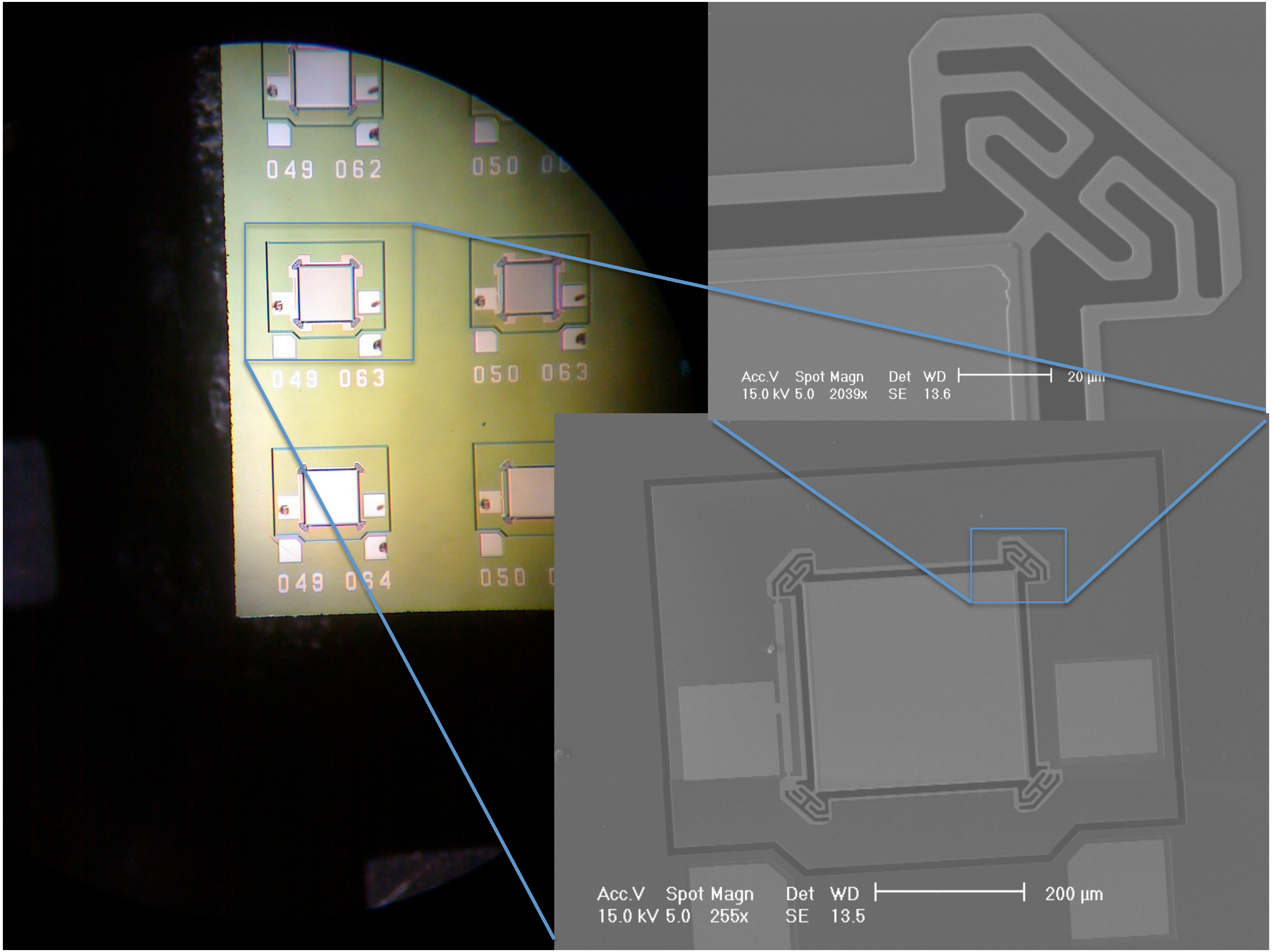

*Figure 3: Electrical connection of the resonator.*

References
[1] V. Kaajakari et al., "Square-Extensional Mode Single-Crystal Silicon Micromechanical Resonator for Low-Phase-Noise Oscillator Applications," *IEEE Electron Device Letters* 25, no. 4 (4, 2004): 173-175.
[2] A. Jaakkola et al., "Piezoelectrically transduced Single-Crystal-Silicon Plate Resonators," in *IEEE Ultrasonics Symposium* (presented at the IEEE Ultrasonics Symposium, Beijing, China, 2008), 2181 – 2184

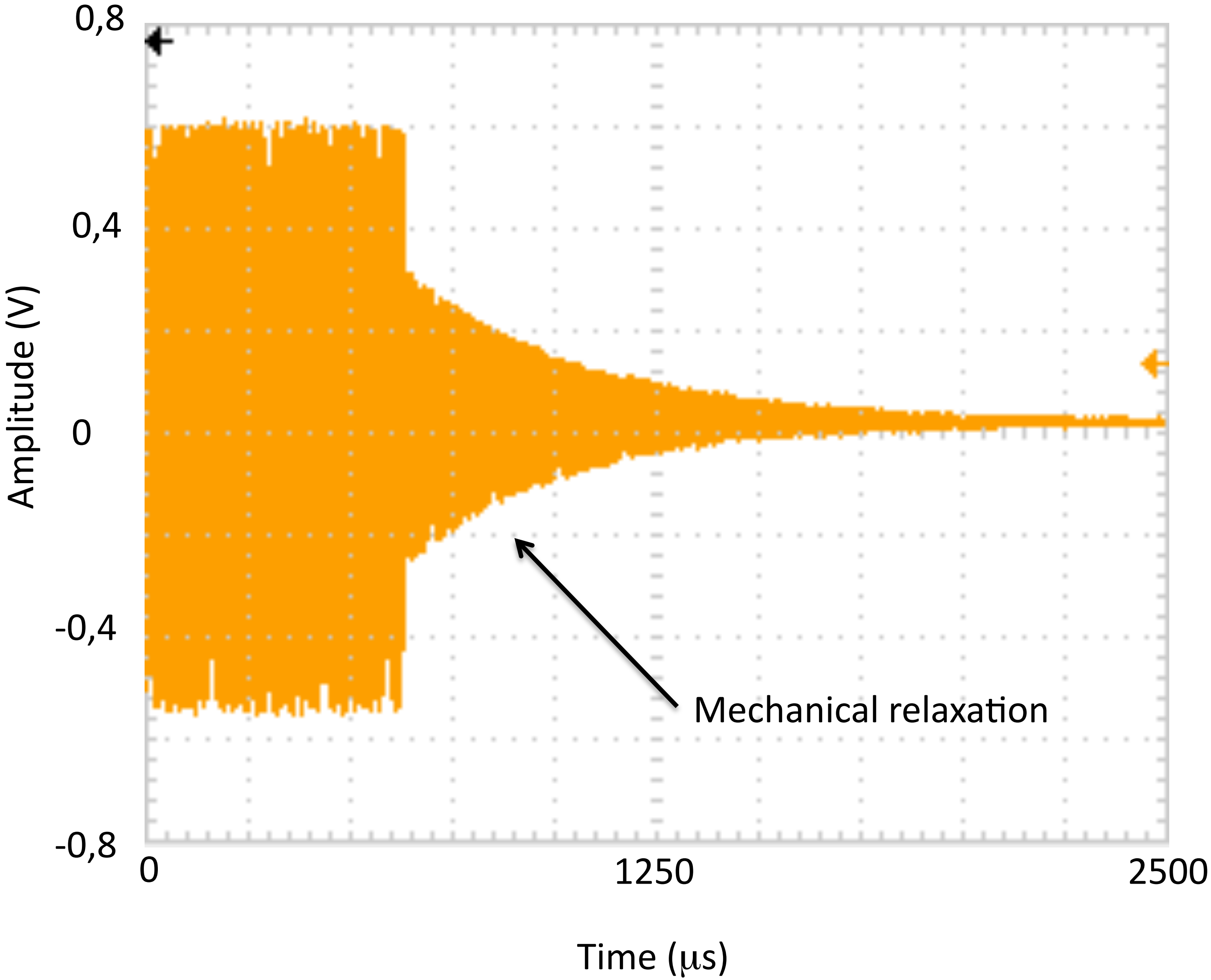



| parameter | value | unit |
|---|---|---|
| Rm | 128.9606 | Ohm |
| Cm | 1.01E-14 | F |
| Lm | 0.013691 | H |
| C0 | 3.00E-11 | F |
| | | |
| f0 | 13.5 | MHz |
| Q | 9000 | 1 |
| k2eff | 0.04 | % |
| | | |
| C1 | <1e-12 | F |
| C2 | <1e-12 | F |

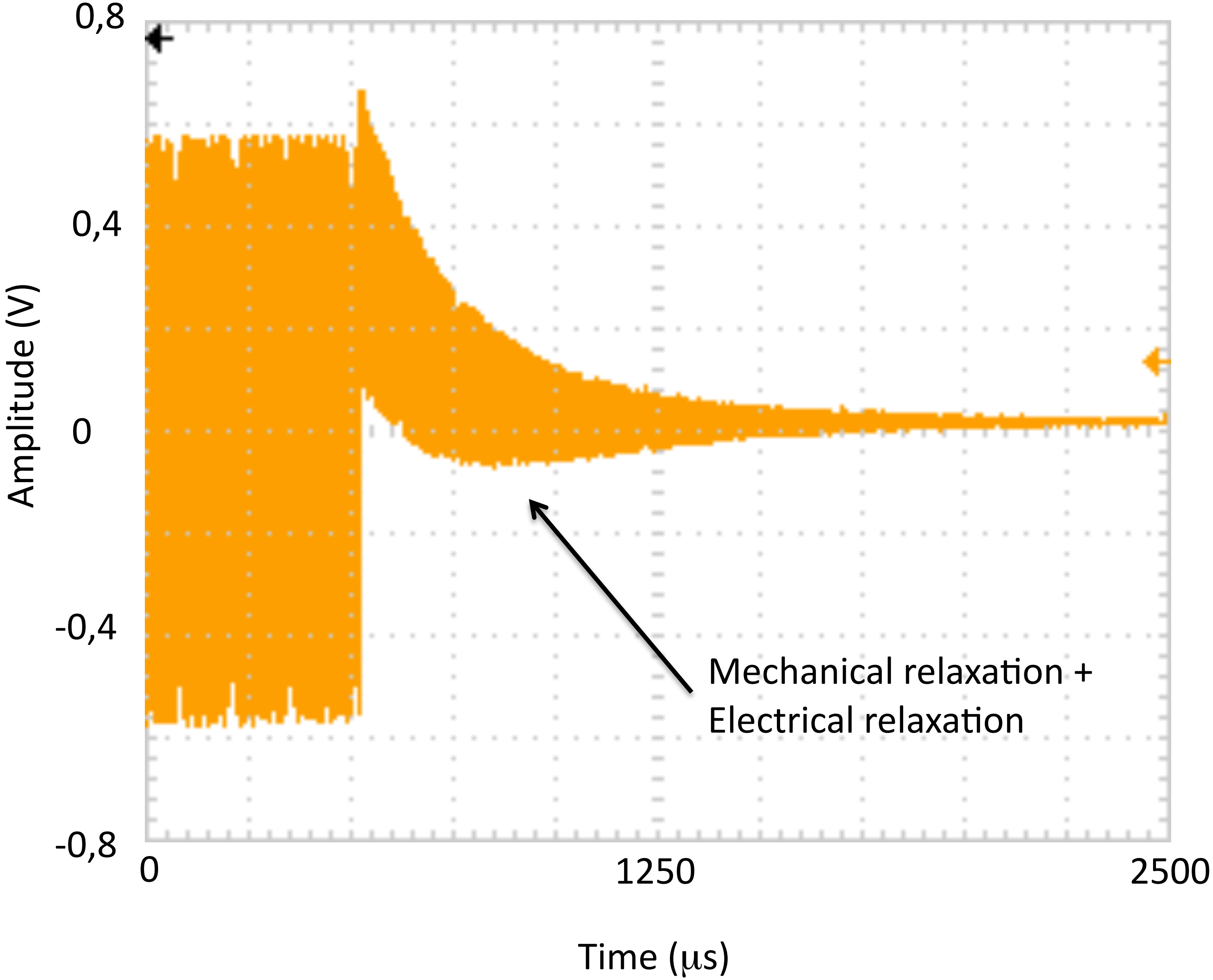
*Table 1: equivalent circuit parameters (and their derivatives).*


049 062
050
049 063
050 063
049 064
050
Acc.V Spot Magn Det WD 20 µm
15.0 kV 5.0 2039x SE 13.6
Acc.V Spot Magn Det WD 200 µm
15.0 kV 5.0 255x SE 13.5

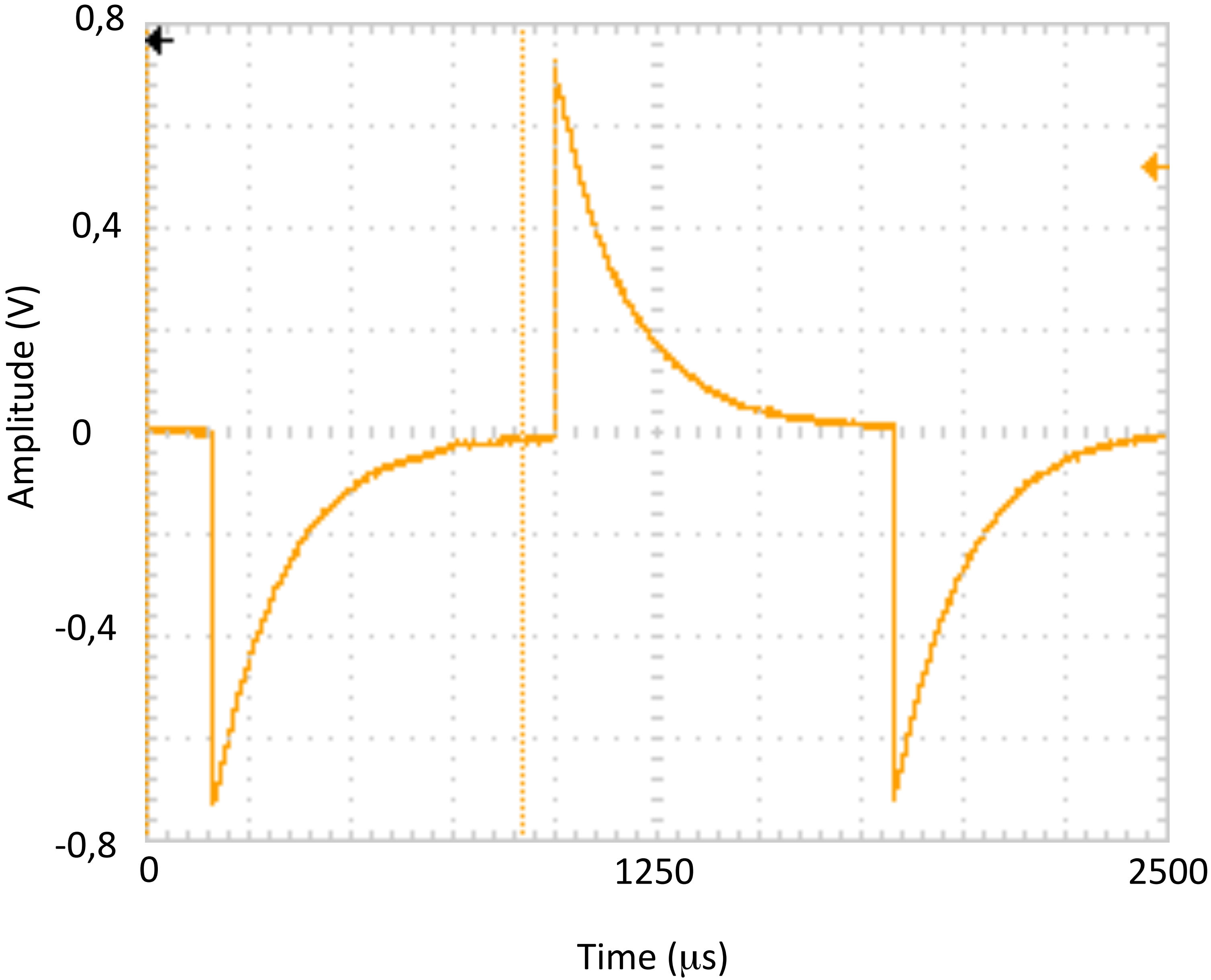
# Voltage sinusoidal excitation switched off at zero



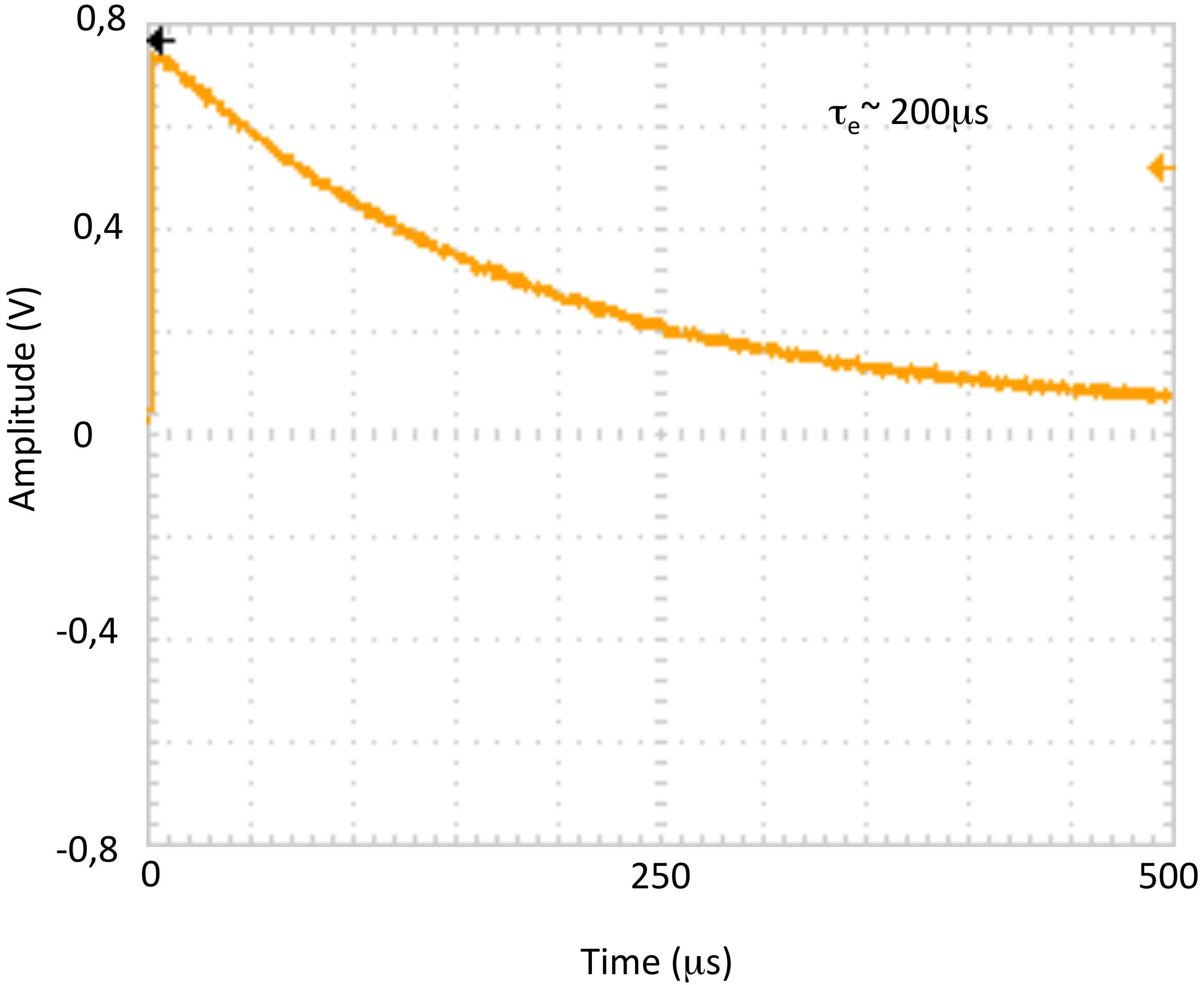
# Voltage sinusoidal excitation switched off at max



# Square wave excitation



# Electrical relaxation time ($\tau_e$)



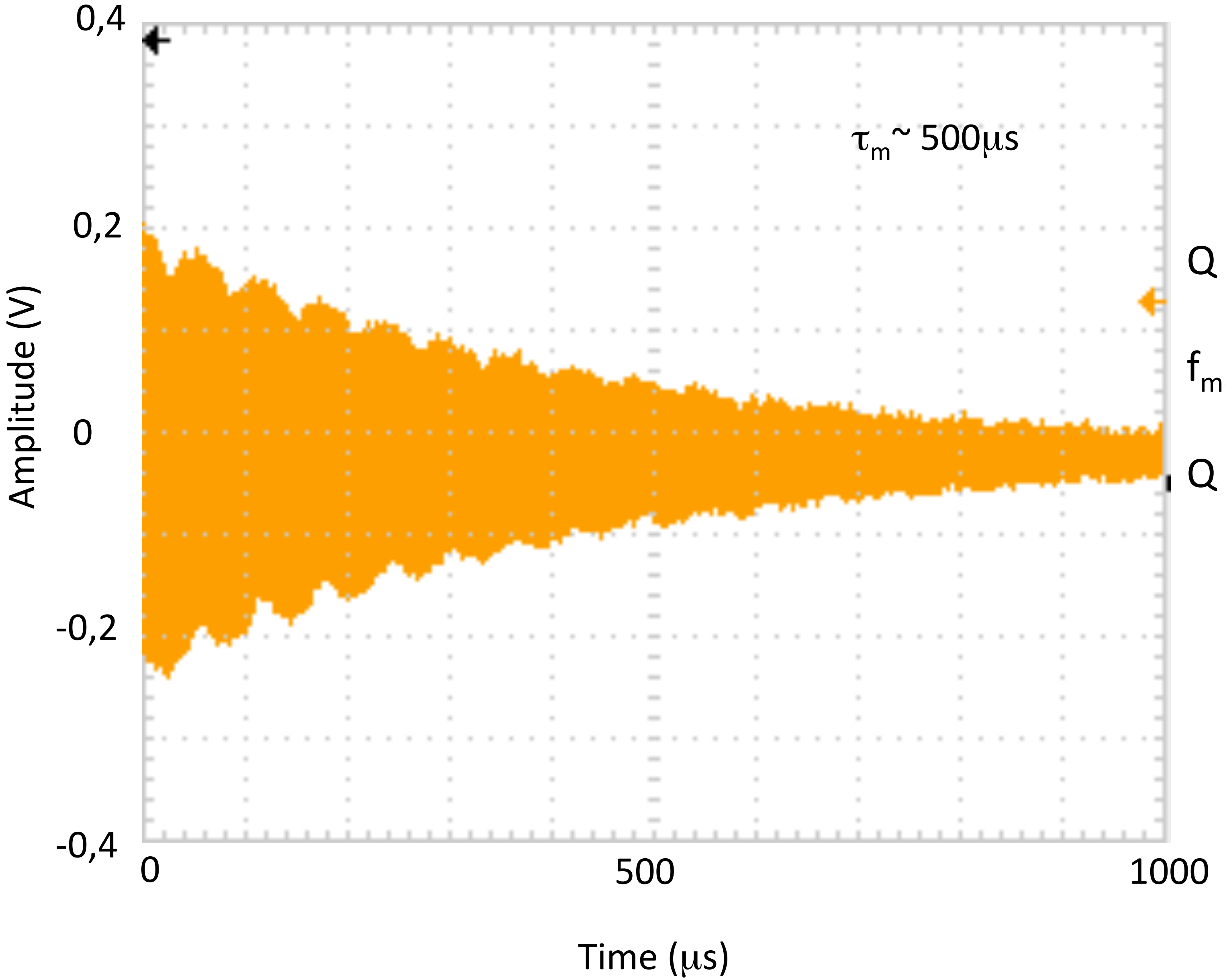

$\tau_e = RC$

$C_{meas.} = 24$ pF

$R_p = 8$ MΩ

# Mechanical relaxation time ($\tau_m$) in vacuum

(open circuit)




$Q = \pi f \tau$

$f_{meas}$ = 13,498 MHz

$Q = 21,2 \cdot 10^3$

# From the model for a linear oscillator:

The voltage transfer function is:

$$|H(\omega)| = \frac{1}{m}\frac{\omega k_c}{\sqrt{\omega^2\left(\frac{k+\frac{\gamma}{\tau}+k_c k_v}{m}-\omega^2\right)^2+\left(\left(\frac{\gamma}{m}+\frac{1}{\tau}\right)\omega^2-\frac{k}{m\tau}\right)^2}}$$

or considering: $\omega_0 = \sqrt{\frac{k+\frac{\gamma}{\tau}+k_c k_v}{m}}$

and: $\omega_1 = \sqrt{\frac{k}{\gamma\tau + m}}$

$$|H(\omega)| = \frac{1}{m}\frac{\omega k_c}{\sqrt{\omega^2\left(\omega^2-\omega_0^2\right)^2+\left(\frac{\gamma\tau+m}{m\tau}\right)^2\left(\omega^2-\omega_1^2\right)^2}}$$

if $\omega^2\left(\omega^2-\omega_0^2\right)^2 > \left(\frac{\gamma\tau+m}{m\tau}\right)^2\left(\omega^2-\omega_1^2\right)^2$

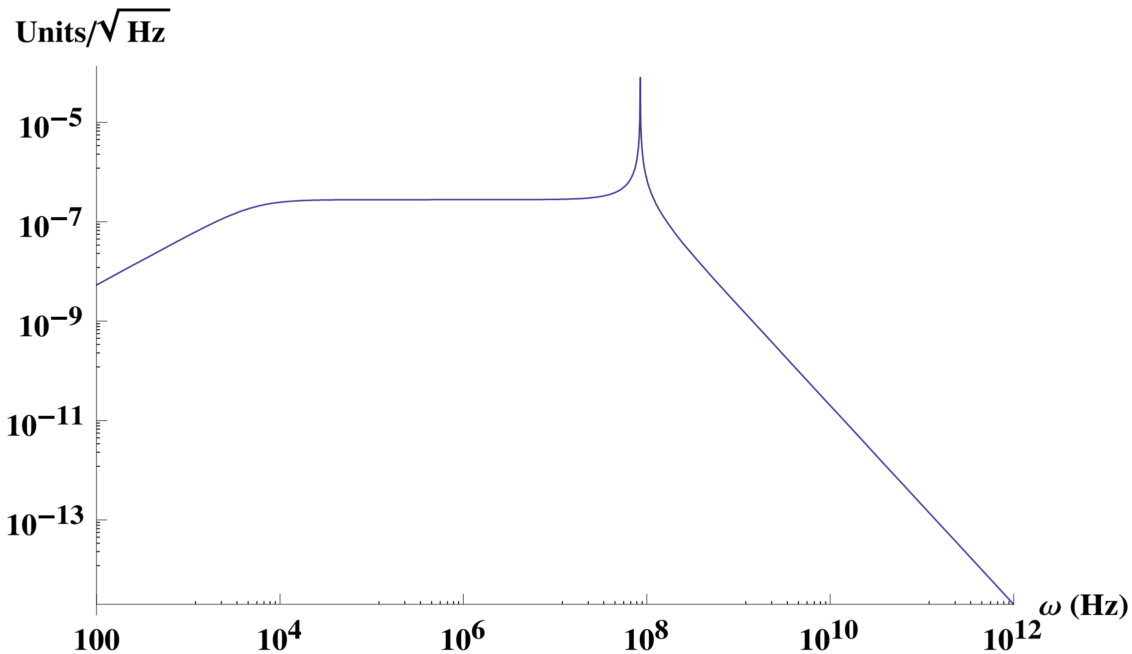
the resonance frequency is $\omega_0 = \sqrt{\frac{k+\frac{\gamma}{\tau}+k_c k_v}{m}}$

where $|H(\omega)|_{\max} = \frac{\omega_0 k_c \tau}{(\gamma\tau + m)\left(\omega_0^2-\omega_1^2\right)^2}$

if $\omega^2\left(\omega^2-\omega_0^2\right)^2 < \left(\frac{\gamma\tau+m}{m\tau}\right)^2\left(\omega^2-\omega_1^2\right)^2$

the resonance frequency is $\omega_1 = \sqrt{\frac{k}{\gamma\tau + m}}$

where $|H(\omega)|_{\max} = \frac{k_c}{m\left|\omega_1^2-\omega_0^2\right|}$



# The analytic result for the Q is:

$$Q = \frac{\omega_r}{\Delta\omega}$$

$\omega_r$ is the resonance frequency and $\Delta\omega$ is the bandwidth (full width when the output voltage is $V_{\max}/\sqrt{2}$ )

$$
\begin{aligned}
\text{Quality Factor} = \Big\{ \Big( & 3\,k\,m\,\tau^2 \sqrt{\Big(2\,m^2 + 2\,(k + \text{kc}\,\text{kv})\,m\,\tau^2 - \gamma^2\,\tau^2 + \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)}\Big)}\Big) \Big/ \\
& \Big(\sqrt{\Big(2\,m^6 + 6\,(k - 2\,\text{kc}\,\text{kv})\,m^5\,\tau^2 + 2\,\gamma^6\,\tau^6 - 2\,\gamma^4\,\tau^4} \\
& \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)} + 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2 \\
& \tau^4\left(-3\,\gamma^2\,\tau^2 + 2\sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)}\right) - \\
& m^4\Big(3\,\gamma^2\,\tau^2 + 18\,\text{kc}\,\text{kv}\,\gamma\,\tau^3 - 6\,k^2\,\tau^4 + 6\,k\,\text{kc}\,\text{kv}\,\tau^4 - 15\,\text{kc}^2\,\text{kv}^2\,\tau^4 + 2 \\
& \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)}\Big) + m^2\,\tau^2\Big(-3\,\gamma^4 \\
& \tau^2 - 18\,\text{kc}\,\text{kv}\,\gamma^3\,\tau^3 + 12\,\text{kc}\,\text{kv}\,\gamma\,\tau \\
& \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)} - 2\,(k + \text{kc}\,\text{kv})^2 \\
& \tau^2\sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)} + \gamma^2\Big(15\,k^2\,\tau^4 \\
& + 30\,k\,\text{kc}\,\text{kv}\,\tau^4 + 15\,\text{kc}^2\,\text{kv}^2\,\tau^4 + 2 \\
& \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)}\Big)\Big) + 2\,m^3\,\tau^2\Big( \\
& k^3\,\tau^4 + 3\,k^2\,\text{kc}\,\text{kv}\,\tau^4 + k\Big(6\,\gamma^2\,\tau^2 + 18\,\text{kc}\,\text{kv}\,\gamma\,\tau^3 + 3\,\text{kc}^2\,\text{kv}^2\,\tau^4 - 2 \\
& \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)}\Big) + \text{kc}\,\text{kv}\Big(-3 \\
& \gamma^2\,\tau^2 + 18\,\text{kc}\,\text{kv}\,\gamma\,\tau^3 + \text{kc}^2\,\text{kv}^2\,\tau^4 + 4 \\
& \sqrt{m^4 + 2\,(k - 2\,\text{kc}\,\text{kv})\,m^3\,\tau^2 - 4\,(k + \text{kc}\,\text{kv})\,m\,\gamma^2\,\tau^4 + \gamma^4\,\tau^4 + m^2\,\tau^2\left(-\gamma^2 - 6\,\text{kc}\,\text{kv}\,\gamma\,\tau + (k + \text{kc}\,\text{kv})^2\,\tau^2\right)}\Big)\Big)\Big)\Big)\Big\}
\end{aligned}
$$

# Considering:

$m$=$10^{-6}$ g

$k$=$7.195\cdot10^{9}$ N/m

$k_v$=2 N/V

$k_c$=2000 V/m

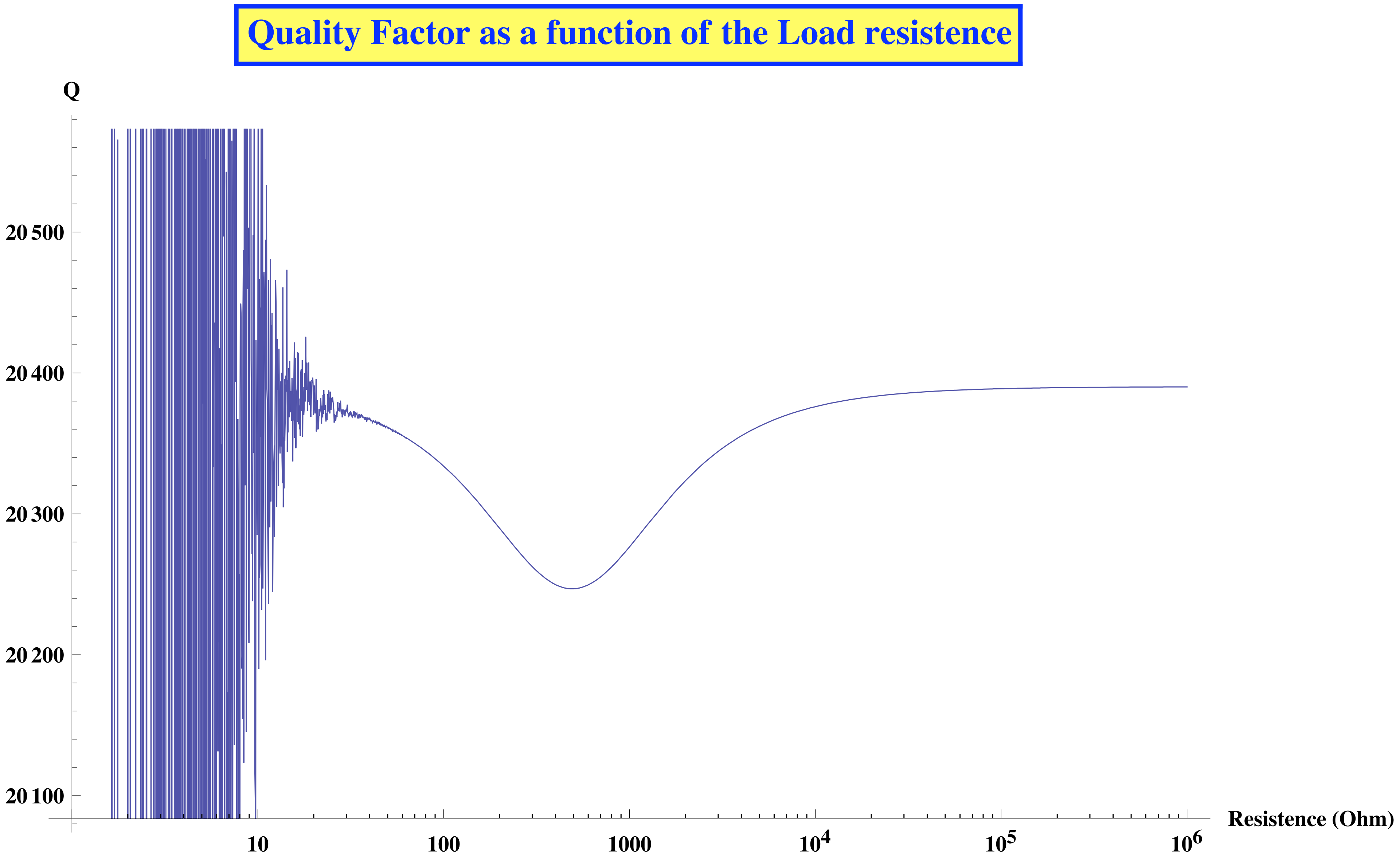
$R_0$=$8\cdot10^{6}$ Ω

$C_0$=$24\cdot10^{-12}$ F

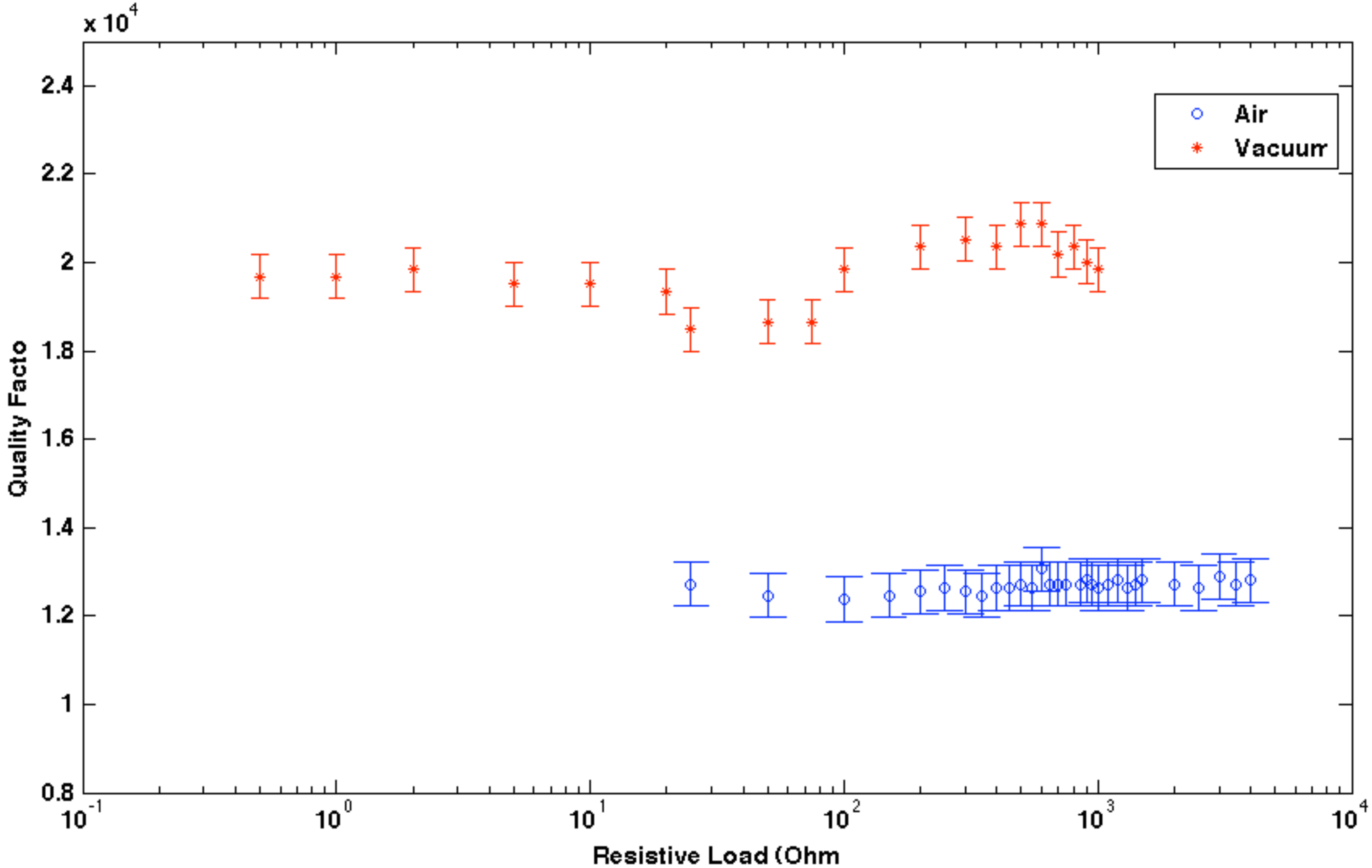
$\gamma$=$m\cdot4.16\cdot10^{3}$ $s^{-1}$

$$K_v = \frac{K_{eff} d_{31} a}{2 t_p k_1}$$

$$K_c = \frac{t_p d_{31} Y_p^E k_1}{a \varepsilon_p}$$


Quality Factor as a function of the Load resistence
Q
20 500
20 400
20 300
20 200
20 100
10
100
1000
10^4
10^5
10^6
Resistence (Ohm)

# Quality Factor vs Resistive load



# Vibrations energy harvesting

Whish list for the perfect vibration harvester

1) Capable of harvesting energy on a broad-band
2) No need for frequency tuning
3) Capable of harvesting energy at low frequency

1) Non-resonant system
2) "Transfer function" with wide frequency resp.
3) Low frequency operated

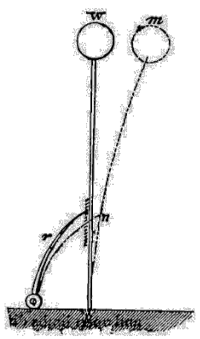
# Noise energy harvesting

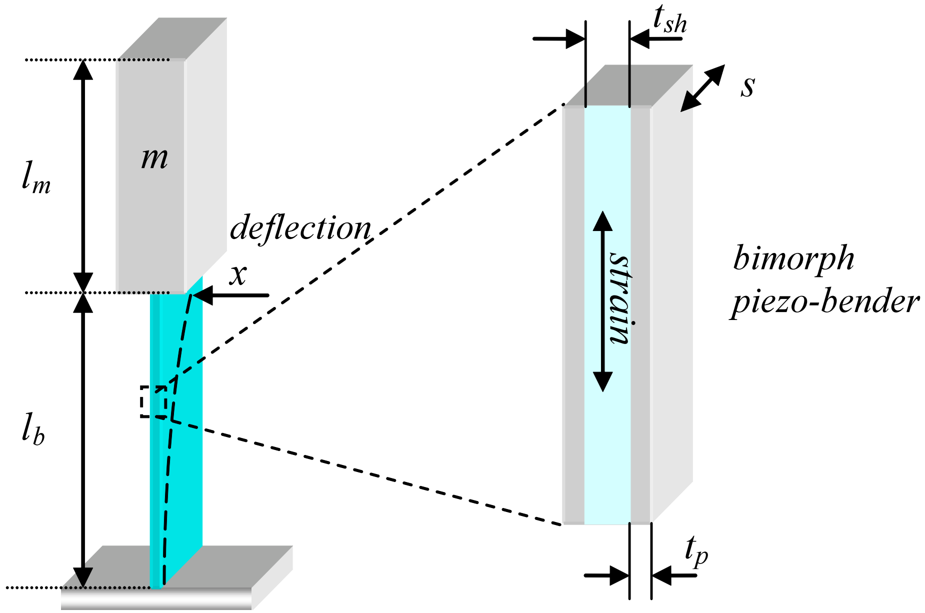
NON-Linear mechanical oscillators

1) Non-resonant system
2) Wide frequency resp.
3) Low frequency operated

Example…

Inverted pendulum






F. Cottone, PhD Thesys, Perugia 2007

# Statistics

- "1D" Statistics: (2nd Order Cumulants, 1st Order Spectra)
  - Correlation: $C_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, d\tau \quad \Leftrightarrow \quad X(f)\, Y^*(f) = S_{xy}(f)$
  - Power Spectral Density: $C_{2x}(t) \quad \Leftrightarrow \quad X(f)\, X^*(f) = S_{2x}(f)$
  - Coherence: $\mathrm{C}_{xy}(f) = \dfrac{S_{xy}(f)}{\sqrt{S_{2x}(f)\, S_{2y}(f)}}$
    - Tells us power and phase coherence at a given frequency

# Statistics (more complicated…)

- "2D" Statistics: (3rd Order Cumulants, 2nd Order Spectra)
  - Bicumulant:

$$C_{xyz}(t,t') = \int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, z(t'+\tau)\, d\tau \quad \Leftrightarrow \quad X(f_1)\, Y(f_2)\, Z^*(f_1+f_2) = S_{xyz}(f_1,f_2)$$

  - Bispectral Density: $C_{3x}(t) \quad \Leftrightarrow \quad X(f_1)\, X(f_2)\, X^*(f_1+f_2) = S_{3x}(f_1,f_2)$

$$S_{3x}(f_1,f_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_{3x}(m,n)\, e^{2\pi i(f_1 m + f_2 n)}\, dm\, dn$$

  - Bicoherence:

$$\mathbf{C}_{xyz}(f) = \frac{S_{xyz}(f_1,f_2)}{\sqrt{S_{xx}(f_1)}\sqrt{S_{yy}(f_2)}\sqrt{S_{zz}(f_1+f_2)}}$$

    - Tells us power and phase coherence at a coupled frequency

# Statistics (more complicated...)

*The Spectrogram* (*STFT square modulus*):

$$S_x(t,\nu) = \left| \int_{-\infty}^{+\infty} x(\tau) h^*(\tau - t) e^{-i2\pi\nu\tau} d\tau \right|^2$$

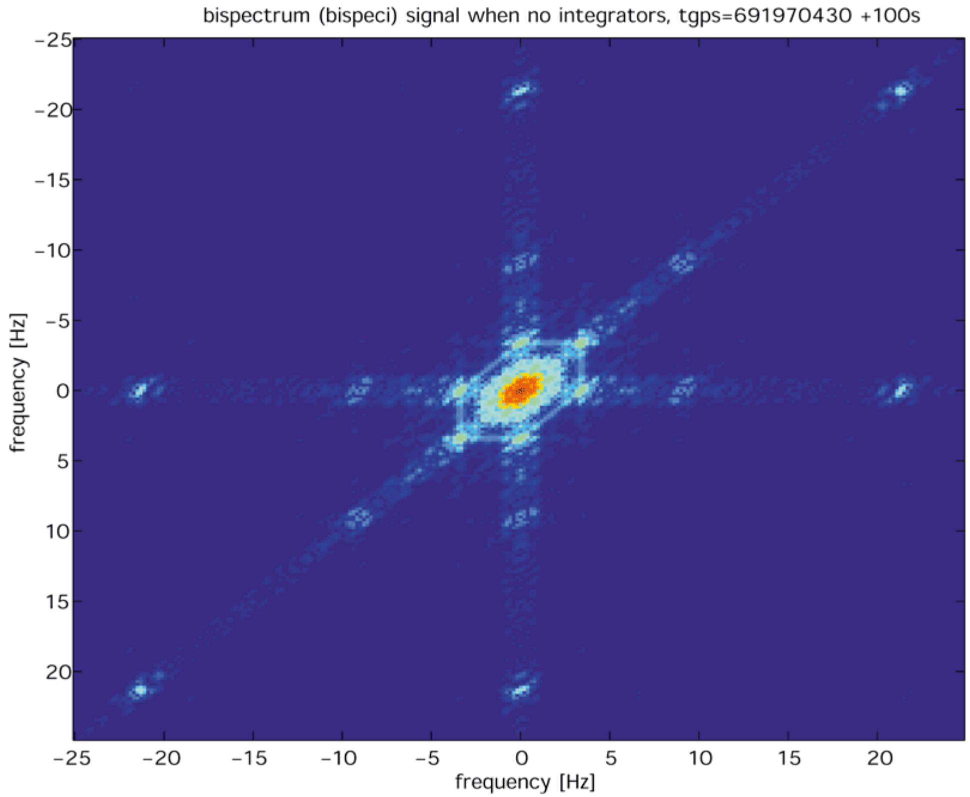
Represents the signal energy in the time-frequency domain centred in (t,$\nu$).

- To analize the system linearity bispectrum and bicoherence need to be taken into account:

- If $S_{3x}=0$ the process is Gaussian and linear

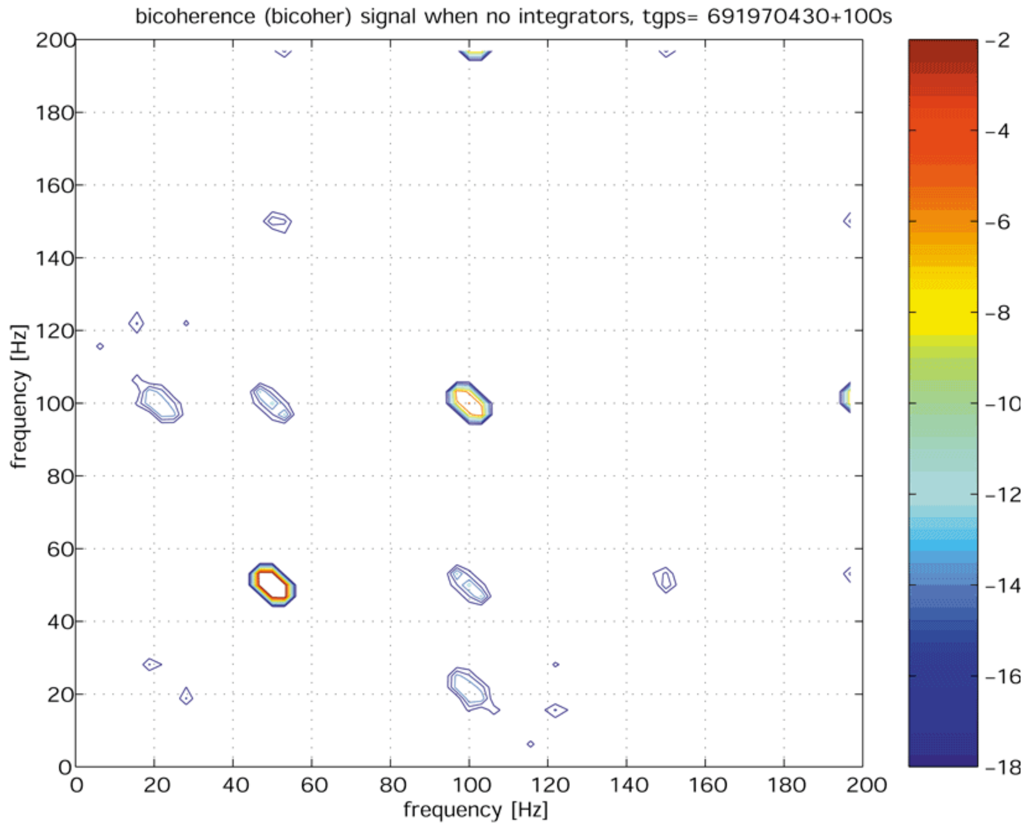
- If $S_{3x}\neq 0$ the process is not Gaussian and

    - if $c_{3x}$ is constant - the process is linear
    - if $c_{3x}$ is not constant - the process is not linear




Bispectrum

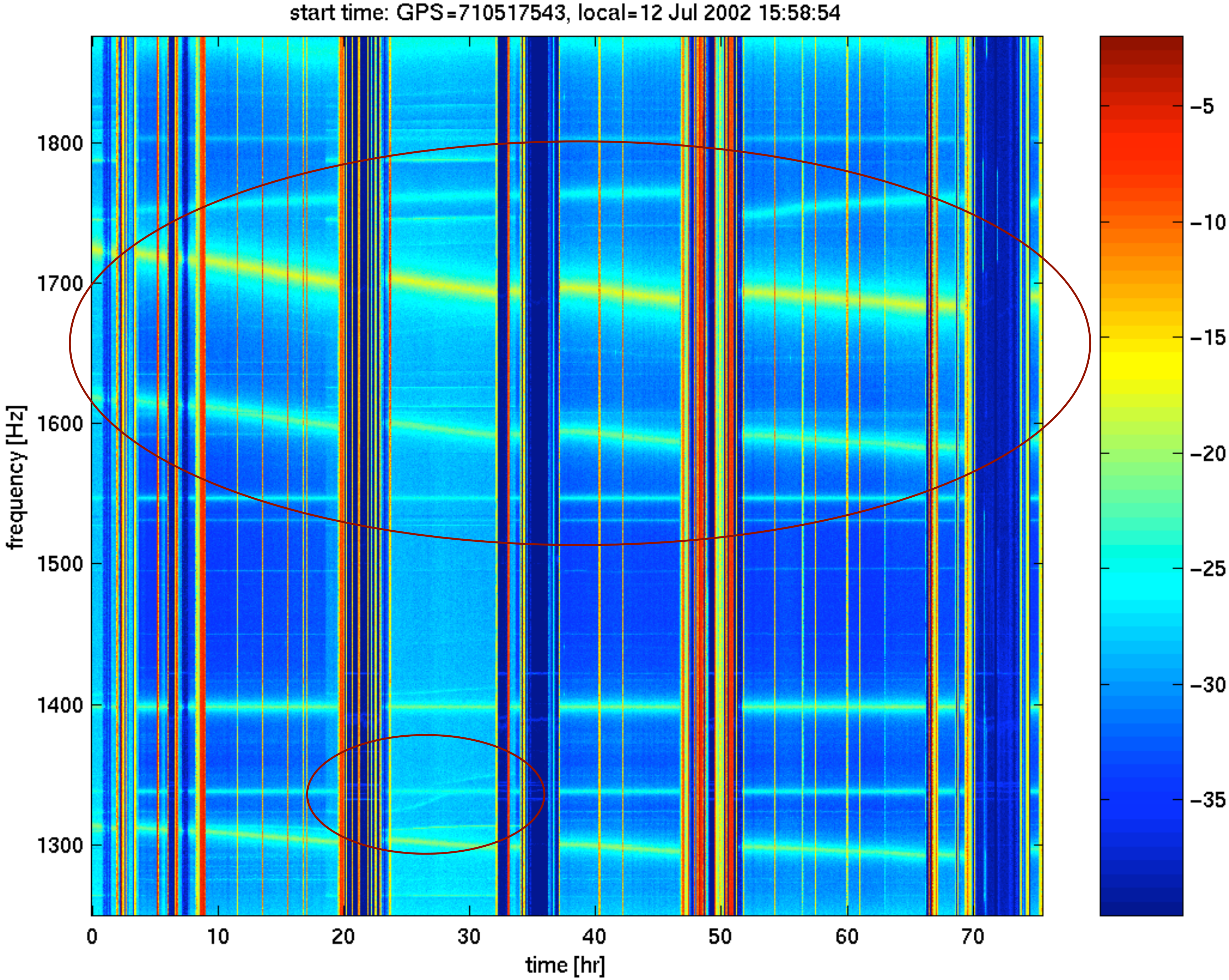
Low frequency noise coupled at higher frequencyes




Bicoherence

A nonlinearity of the 50 Hz with its armonics is observed.
There is present a big coupling between the 20 Hz and the 100 Hz and a smaller one between the 20 and 30 Hz.

Spectrogram:



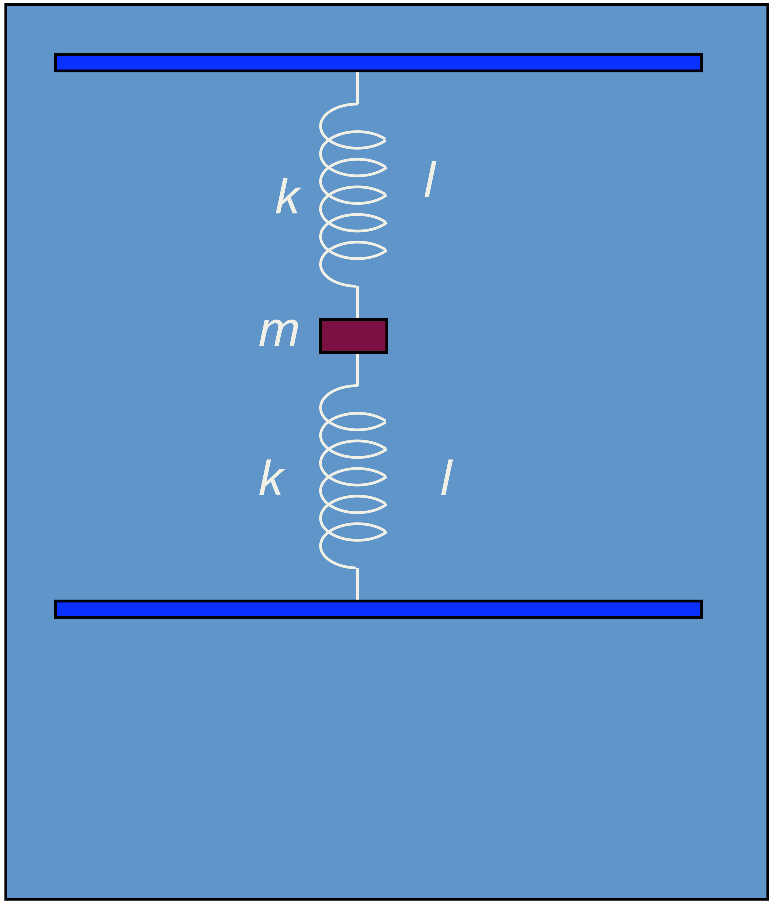
Let's look at an example of non-linear oscillator:

the Duffing Oscillator

$$\ddot{x} + \delta \dot{x} + \beta x + \alpha x^3 = \gamma \cos \omega t$$

$$U(x) = \frac{1}{2}\beta x^2 + \frac{1}{4}\alpha x^4$$

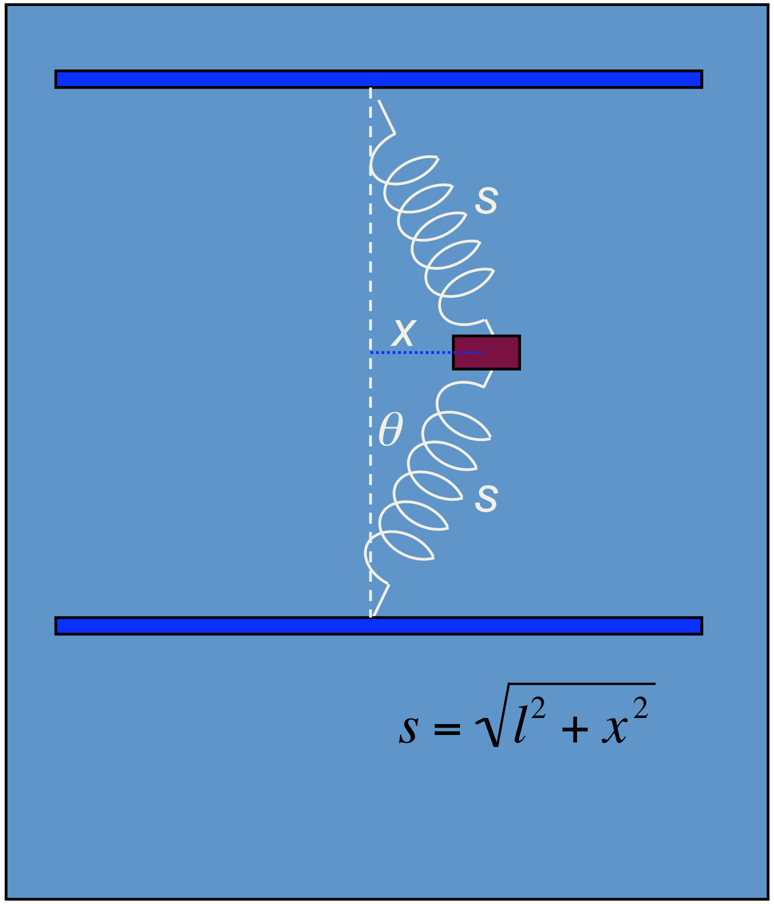
# A two springs system




- A mass is held between two springs.
  - Spring constant *k*
  - Natural length *l*

- Springs are on a horizontal surface.
  - Frictionless
  - No gravity

# Transverse Displacement




- The force for a displacement is due to both springs.
  - Only transverse component
  - Looks like its harmonic

$$F = -2k\left(\sqrt{l^2 + x^2} - l\right)\sin\theta$$

$$= -2k\left(\sqrt{l^2 + x^2} - l\right)\frac{x}{\sqrt{l^2 + x^2}}$$

$$= -2kx\left(1 - \frac{1}{\sqrt{1 + x^2/l^2}}\right)$$

# Purely Nonlinear

- The force can be expanded as a power series near equilibrium.
  - Expand in *x*/*l*

$$F = -2kl\frac{x}{l}\left(1 - \frac{1}{\sqrt{1 + x^2/l^2}}\right)$$

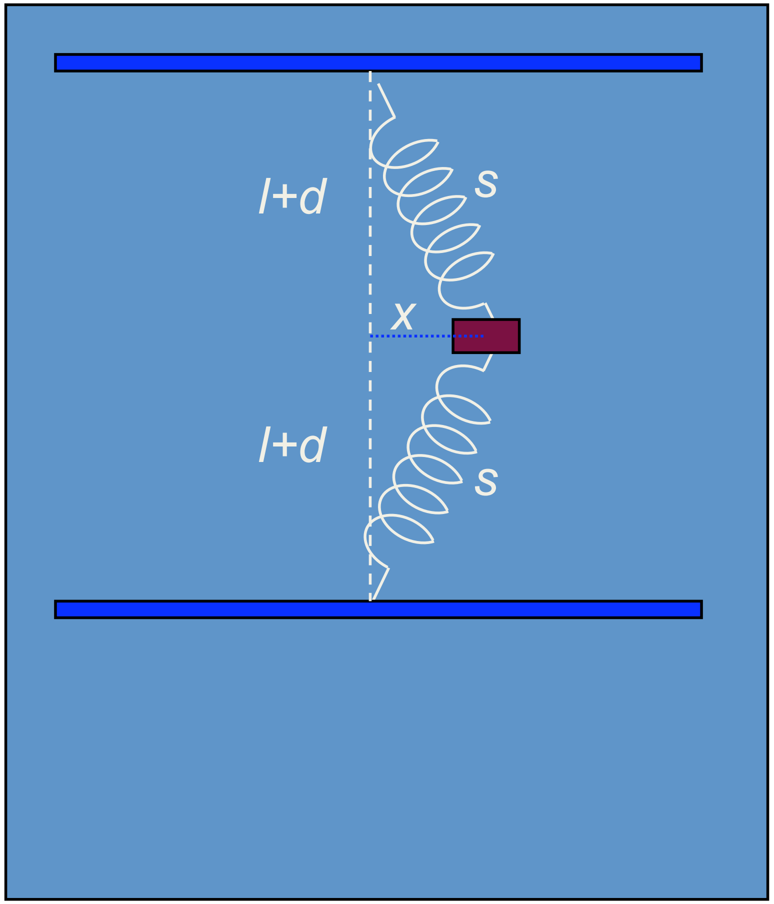
- The lowest order term is non-linear.

$$F \cong -kl\left(\frac{x}{l}\right)^3 + \ldots$$

- Quartic potential
  - Not just a perturbation

$$V \cong \frac{k}{4l^2}x^4 + \ldots$$

# Mixed Potential



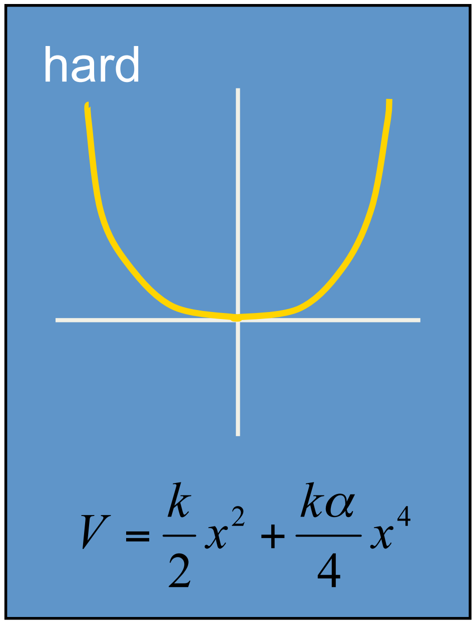

- Typical springs are not at natural length.
  - Approximation includes a linear term

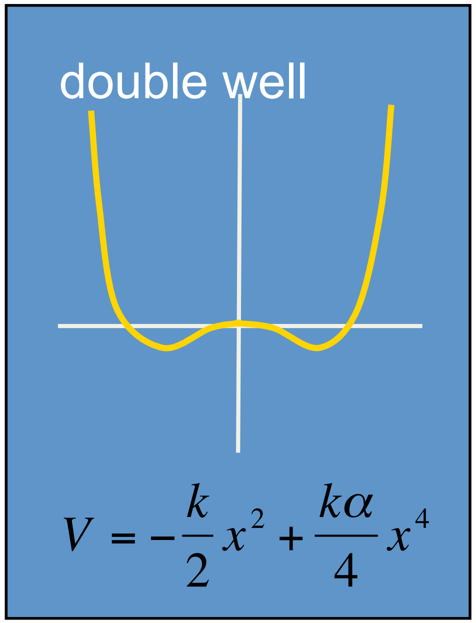
$$F \cong -\frac{2kd}{l}x - \frac{k(l-d)}{l^3}x^3 + \ldots$$

$$V \cong \frac{kd}{l}x^2 + \frac{k(l-d)}{4l^3}x^4 + \ldots$$

# Quartic Potentials

- The sign of the forces influence the shape of the potential.



# Driven System

- Assume a more complete, realistic system.
  - Damping term
  - Driving force

$$m\ddot{x} = -\beta\dot{x} - kx - k\alpha x^3 + f\cos\omega t$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + \alpha\omega_0^2 x^3 = f\cos\omega t$$

- Rescale the problem:
  - Set $t$ such that $\omega_0^2 = k/m = 1$
  - Set $x$ such that $k\alpha/m = 1$

- This is the Duffing equation

$$\ddot{x} + \gamma\dot{x} + x + x^3 = f\cos\omega t$$

# Steady State Solution

- Try a solution, match terms

$$x(t) = A(\omega)\cos[\omega t - \theta(\omega)]$$

$$\ddot{x} + \gamma\dot{x} + x + x^3 = f\cos\omega t$$

$$A(1-\omega^2)\cos(\omega t - \theta) - A\gamma\omega\sin(\omega t - \theta) + A^3\cos^3(\omega t - \theta) = f\cos\omega t$$

trigonometric identities

$$\cos^3(\omega t - \theta) = \tfrac{3}{4}\cos(\omega t - \theta) + \tfrac{1}{4}\cos 3(\omega t - \theta)$$

$$f\cos\omega t = f\cos\theta\cos(\omega t - \theta) - f\sin\theta\sin(\omega t - \theta)$$

$$[A(1-\omega^2 + \tfrac{3}{4}A^2) - f\cos\omega t]\cos(\omega t - \theta)$$
$$+[-A\gamma\omega + f\sin\omega t]\sin(\omega t - \theta)$$
$$+\tfrac{1}{4}A^3\cos 3(\omega t - \theta)$$
$$= 0$$

$$f\cos\omega t = A(1-\omega^2 + \tfrac{3}{4}A^2)$$
$$f\sin\omega t = A\gamma\omega$$
$$\tfrac{1}{4}A^3\cos 3(\omega t - \theta) \approx 0$$

# Amplitude Dependence

- Find the amplitude-frequency relationship.
  - Reduces to forced harmonic oscillator for A → 0

$$f^2 = A^2[(1-\omega^2)^2 + (\gamma\omega)^2]$$

$$f^2 \cos^2 \omega t = A^2(1-\omega^2 + \tfrac{3}{4}A^2)^2$$

$$f^2 \sin^2 \omega t = A^2\gamma^2\omega^2$$

$$f^2 = A^2[(1-\omega^2 + \tfrac{3}{4}A^2)^2 + \gamma^2\omega^2]$$

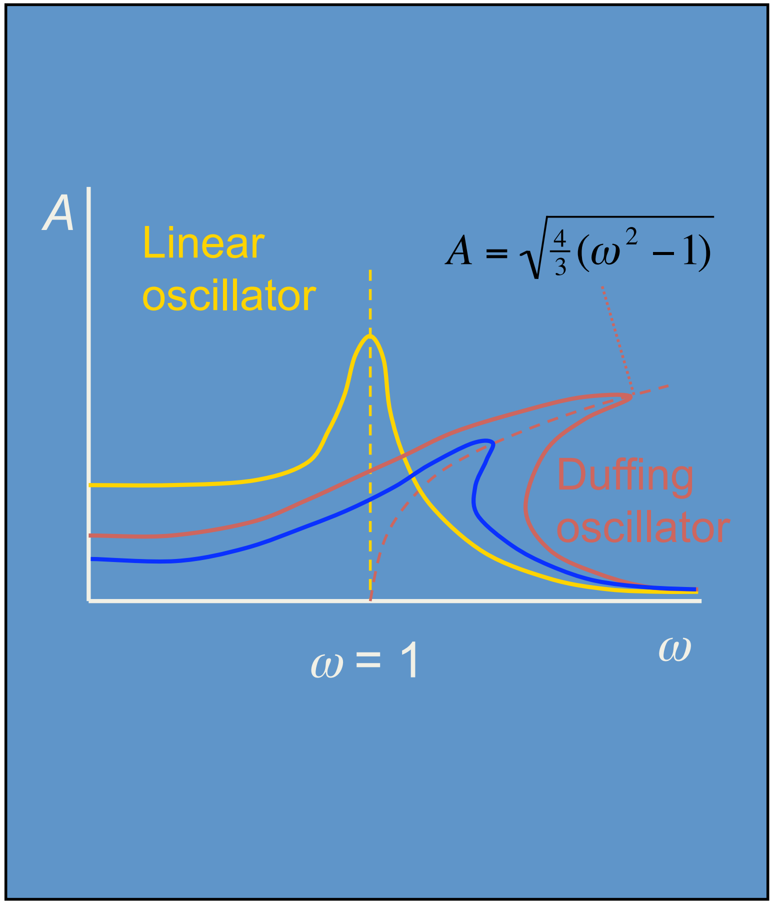
- Find the case for minimal damping and driving force.
  - *f*, $\gamma$ both near zero
  - Defines resonance condition

$$0 = A^2[(1-\omega^2 + \tfrac{3}{4}A^2)^2 + 0]$$

$$0 = 1-\omega^2 + \tfrac{3}{4}A^2$$

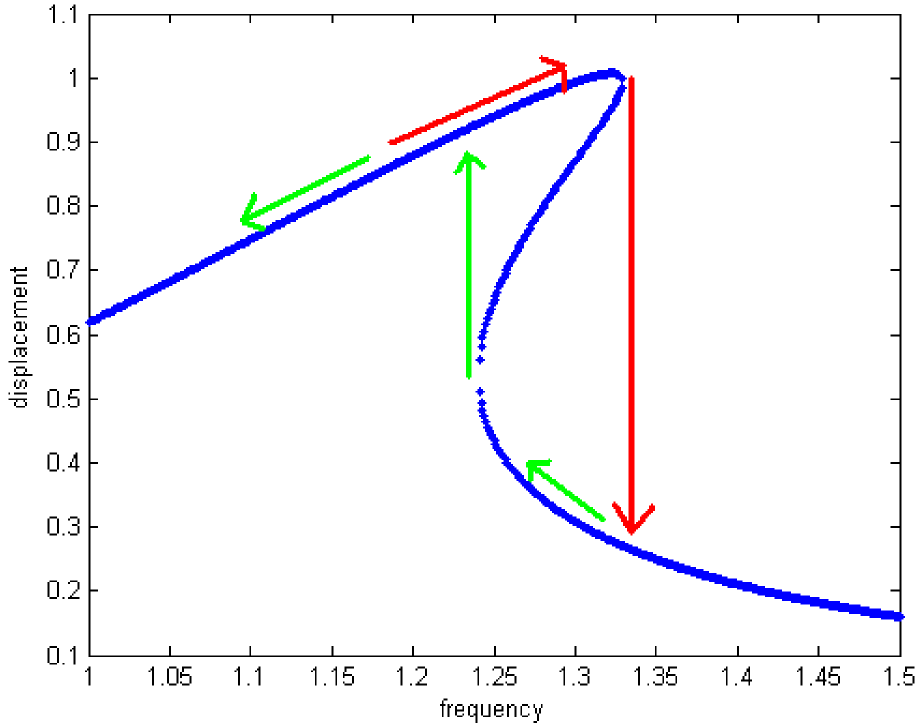
$$A(\omega) = \sqrt{\tfrac{4}{3}(\omega^2 - 1)}$$

# Nonlinear Resonance Frequency




- The resonance frequency of a linear oscillator is independent of amplitude.
- The resonance frequency of a Duffing oscillator increases with amplitude.

# … brings to hysteresis




$\omega_1$ $\omega_2$

- A Duffing oscillator behaves differently for increasing and decreasing frequencies.
  - Increasing frequency has a jump in amplitude at $\omega_2$
  - Decreasing frequency has a jump in amplitude at $\omega_1$

- This is hysteresis.

# Nonlinear Resonance

(in general…)

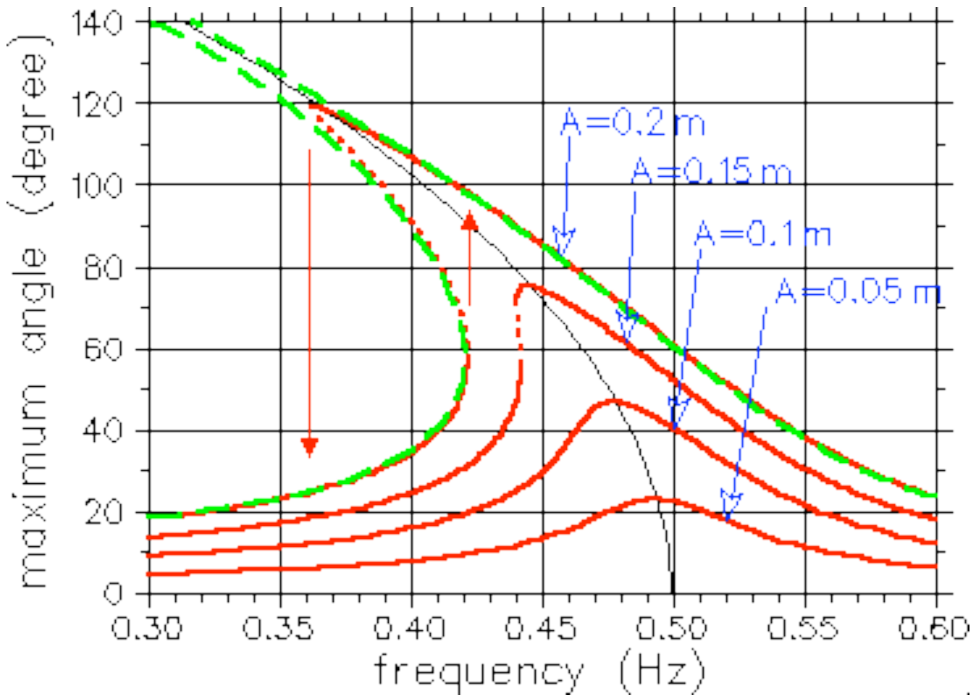
Nonlinear resonance seems not to be so much different from the (linear) resonance of a harmonic oscillator. But both, the dependency of the eigenfrequency of a nonlinear oscillator on the amplitude and the nonharmoniticity of the oscillation lead to a behavior that is impossible in harmonic oscillators, namely, the foldover effect and superharmonic resonance, respectively.

Both effects are especially important in the case of weak damping.

# The foldover effect

The foldover effect got its name from the bending of the resonance peak in a amplitude versus frequency plot. This bending is due to the frequency-amplitude relation which is typical for nonlinear oscillators.

$$g = 9.81 m/\sec^2, l = 1m, \gamma = 0.4 \sec^{-1}$$




Foldover effect for a pendulum

The pendulum eq.:

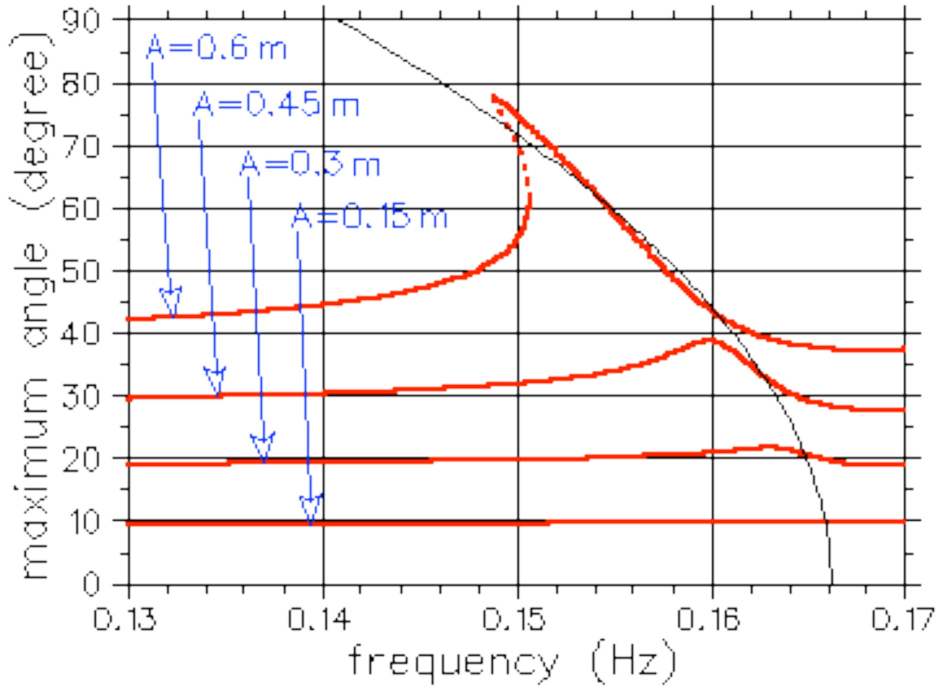
$$\ddot{\varphi} = -\gamma\dot{\varphi} - \omega_0^2 \sin\varphi + f \cos\omega t$$

$$\omega_0^2 = \frac{g}{l}$$

# The superharmonic resonance

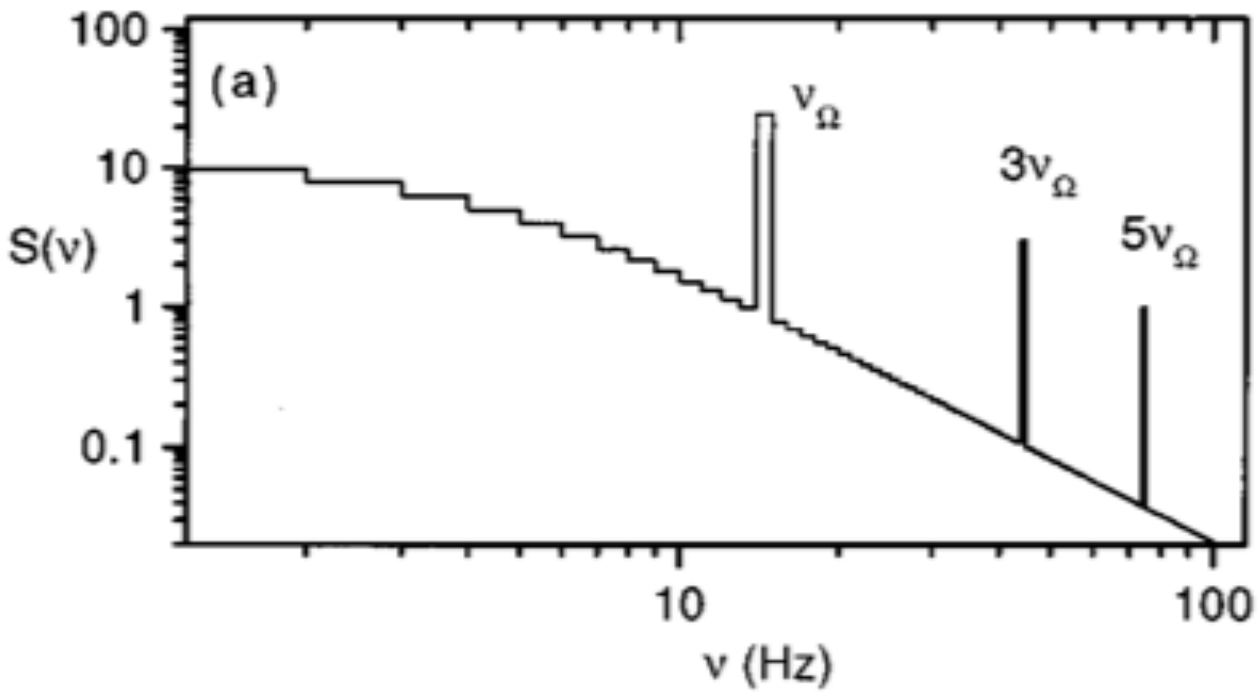
Nonlinear oscillators do not oscillate sinusoidal.
Superharmonic resonance is simply the resonance with one of this higher harmonics of a nonlinear oscillation. In an amplitude/frequency plot appear additional resonance peaks. In general, they appear at driving frequencies which are integer fractions of the fundamental frequency.

$$g = 9.81 m/\sec^2, l = 1m, \gamma = 0.1 \sec^{-1}$$



# Bistable Duffing

In case of a bistable oscillator the frequency response for an overdamped system is highly spread in the low frequency region.

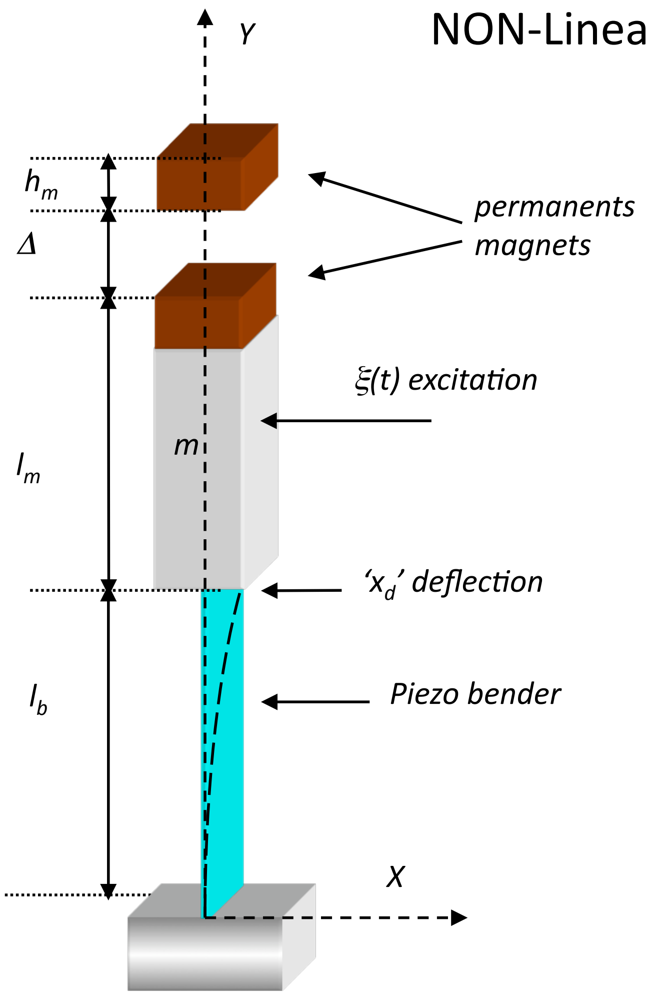


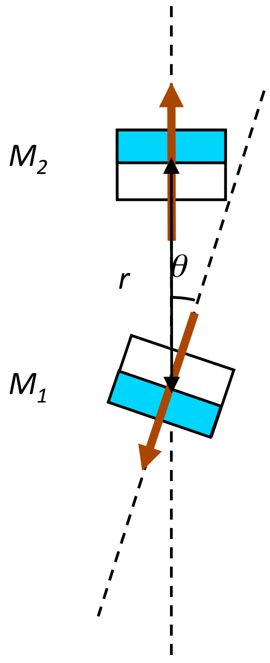

Gammaitoni et al. Reviews of Modern Physics 1998

# Noise energy harvesting

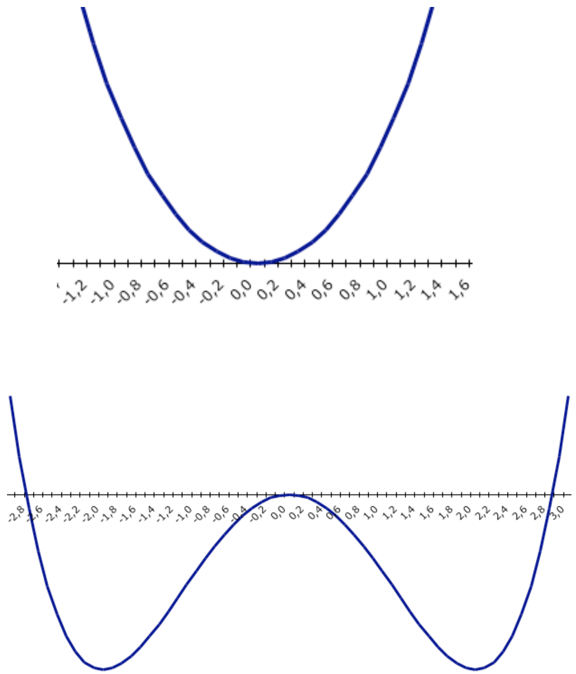
NON-Linear mechanical oscillators

NON-Linear Inverted pendulum



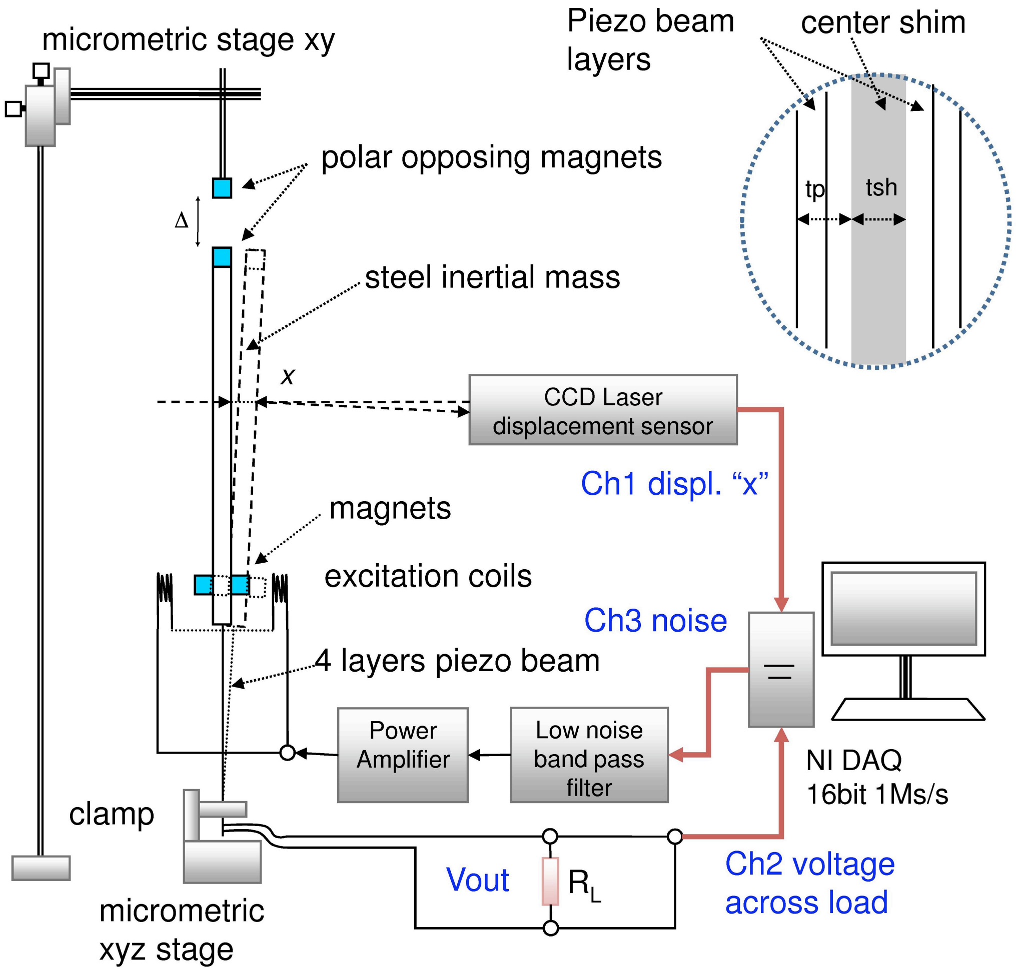




b)



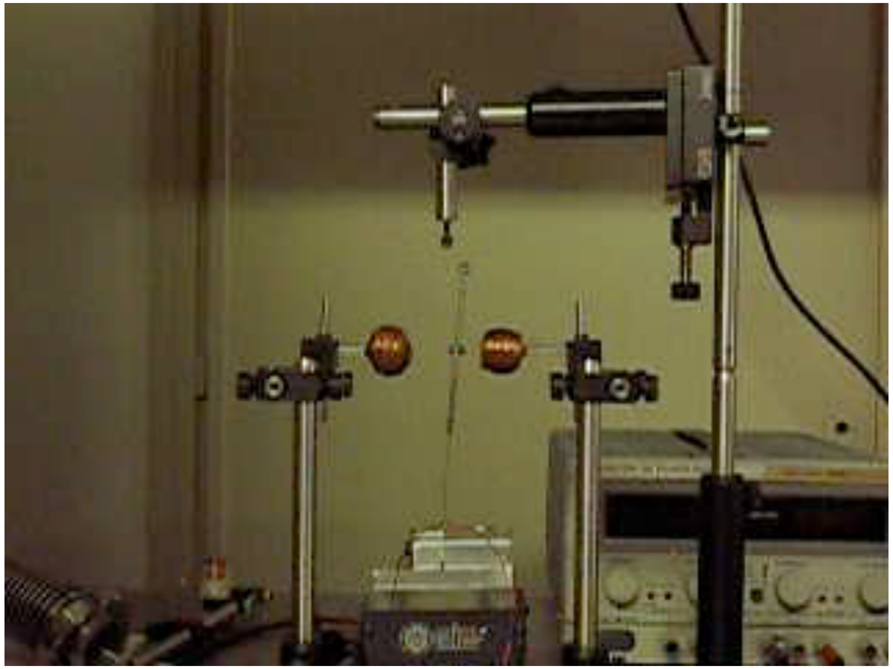
# Noise energy harvesting

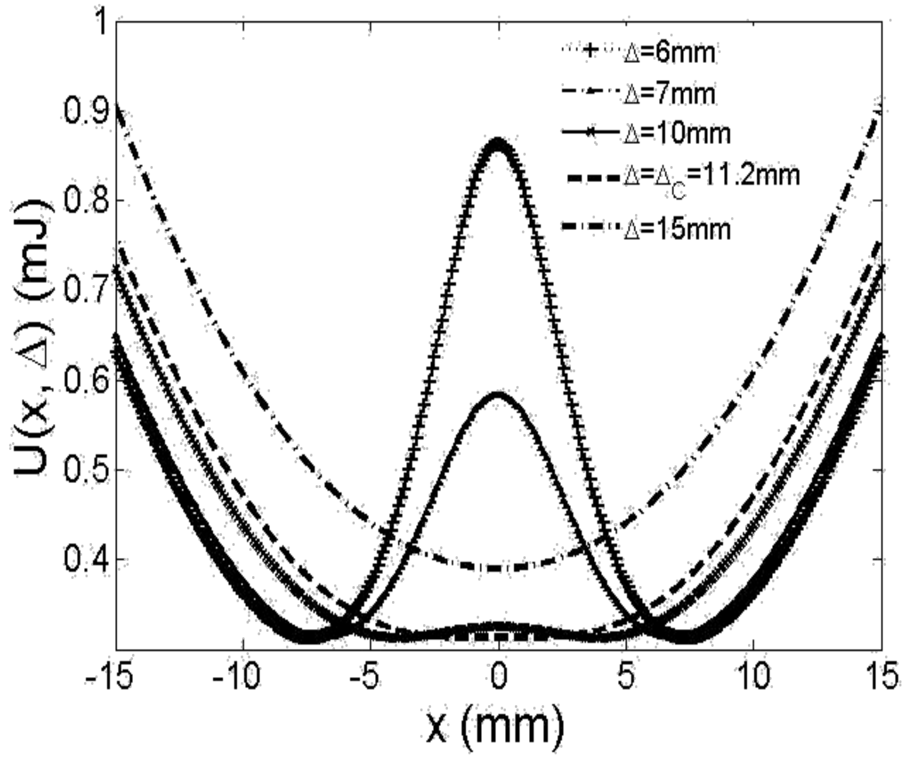
NON-Linear mechanical oscillators



# Noise energy harvesting

NON-Linear mechanical oscillators



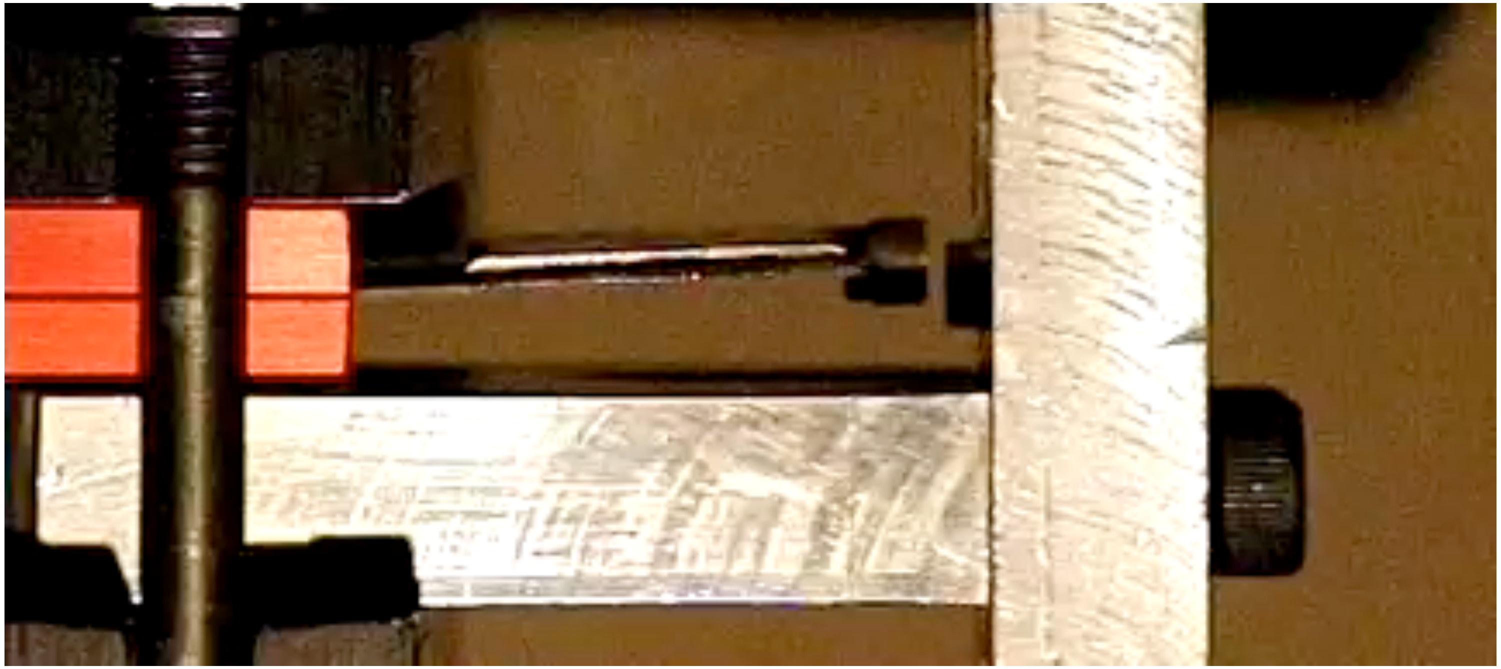



http://www.nipslab.org/node/1676

Nonlinear Energy Harvesting, F. Cottone; H. Vocca; L. Gammaitoni
**Physical Review Letters**, 102, 080601 (2009)

# Noise energy harvesting

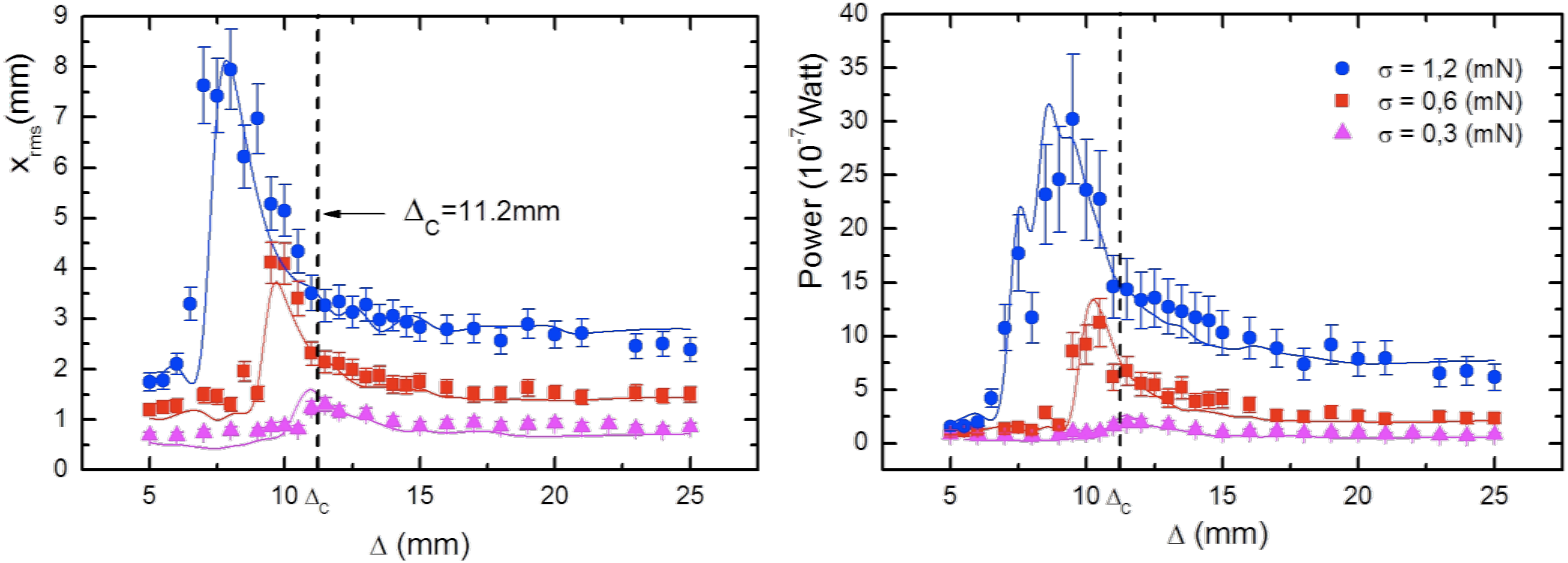
NON-Linear mechanical oscillators



Nonlinear Energy Harvesting, F. Cottone; H. Vocca; L. Gammaitoni
**Physical Review Letters**, 102, 080601 (2009)

# Noise energy harvesting

NON-Linear mechanical oscillators



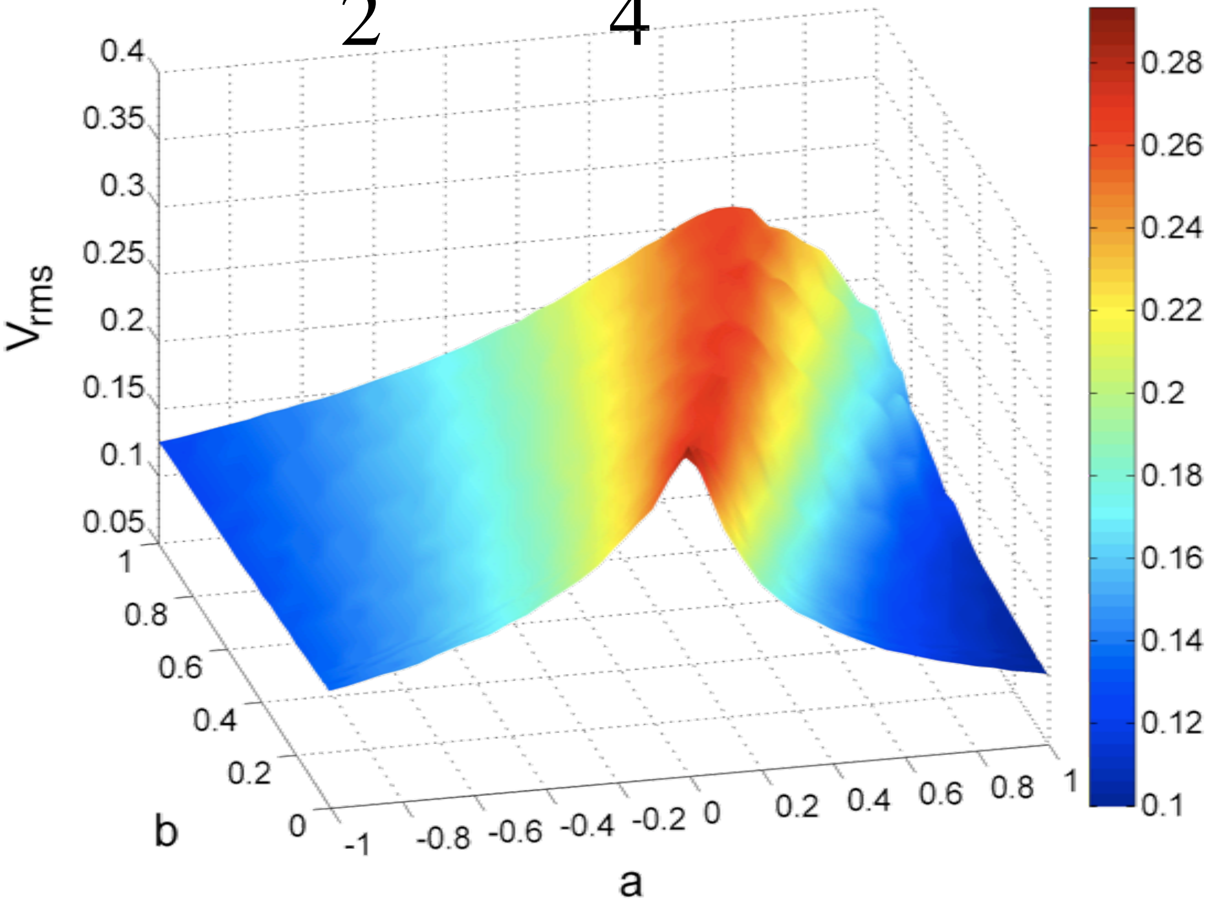

Nonlinear Energy Harvesting, F. Cottone; H. Vocca; L. Gammaitoni , Physical Review Letters, 102, 080601 (2009)

# Noise energy harvesting

Non-linear systems

$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

Duffing potential

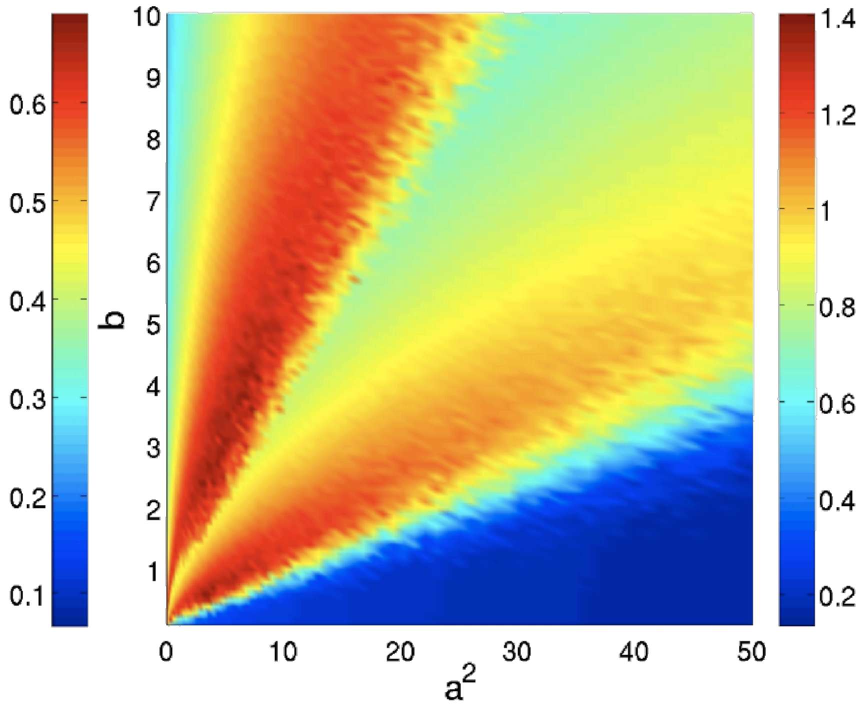



L. Gammaitoni, I. Neri, H. Vocca, Appl. Phys. Lett. 94, 164102 (2009)

# Noise energy harvesting

Non-linear systems

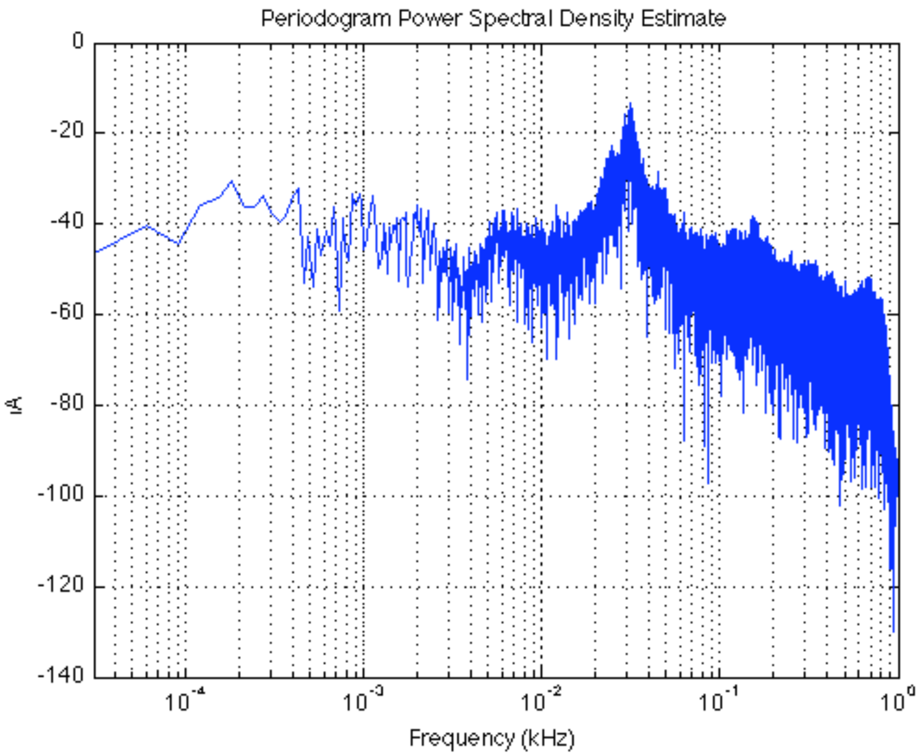
$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$ Duffing potential

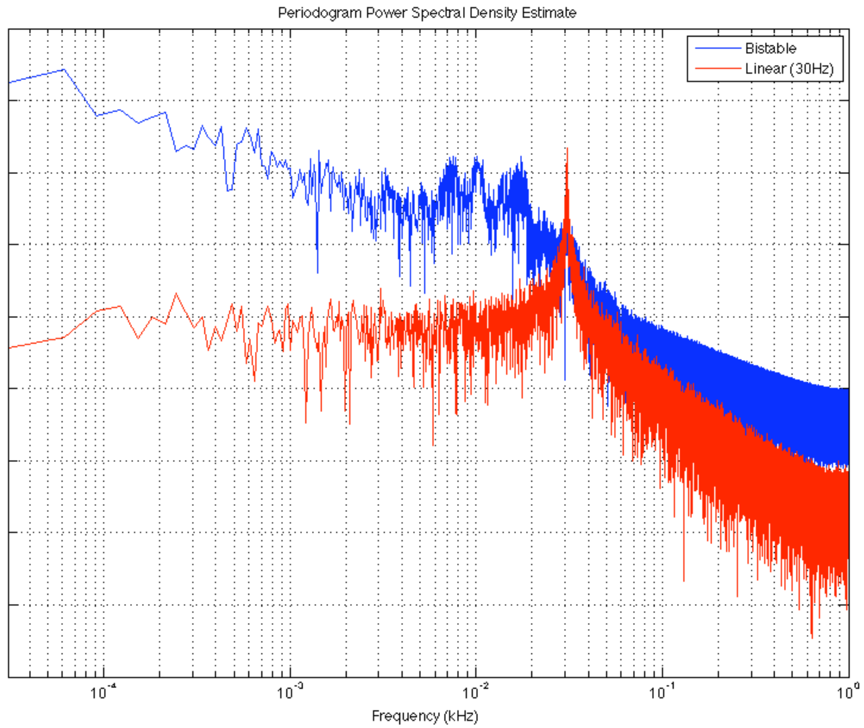


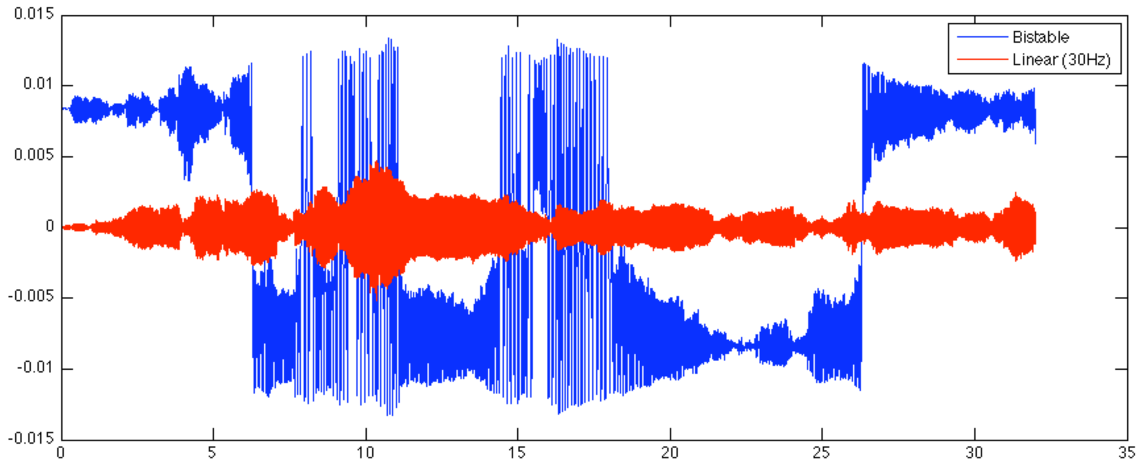

$$b_{MAX} = \frac{a^2}{4D\ \log(\tau_p)}$$

L. Gammaitoni, I. Neri, H. Vocca, Appl. Phys. Lett. 94, 164102 (2009)

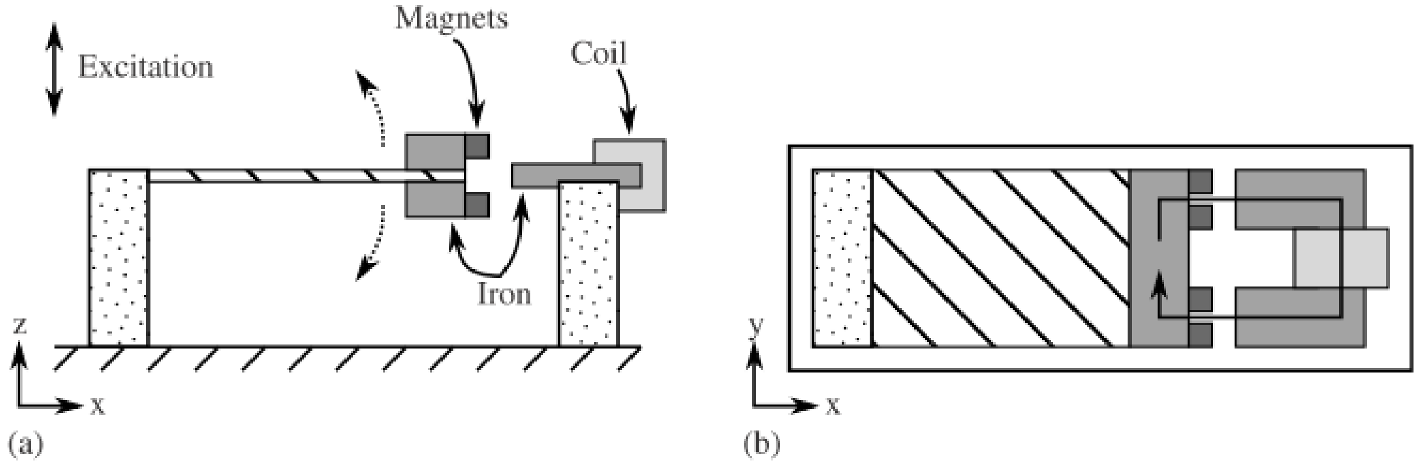
# A compared response



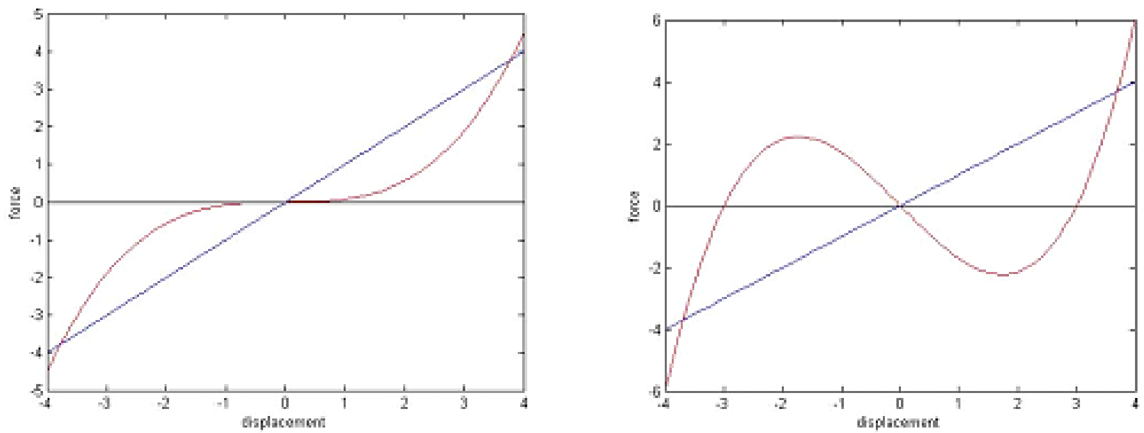



Barton D.A.W., Burrow S.G. and Clare L.R., 2010, "Energy Harvesting from Vibrations with a Nonlinear Oscillator," *Journal of Vibration and Acoustics*, 132, 021009.




$$m\ddot{x} + (c + \frac{\theta^2}{R_C + R_L})\dot{x} + kx + \beta x^3 = F\sin(\omega t)$$

*k < 0* → hardening
*k > 0* → bistable




After non-dimensionalization: $$\ddot{x} + 2\xi_{eff}\dot{x} + x + \beta x^3 = \Gamma\sin(\omega t)$$

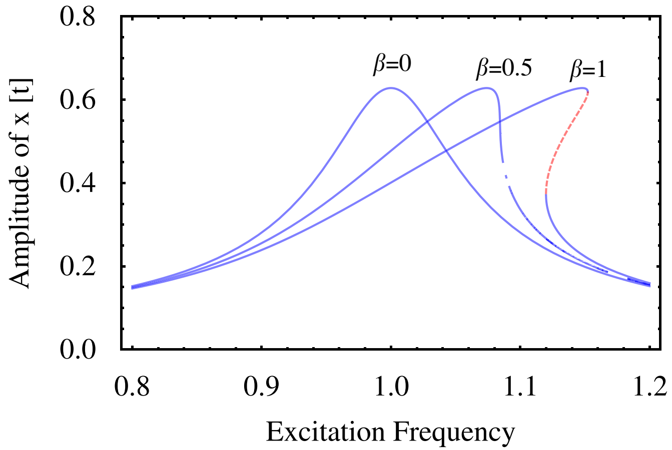
# Noise energy harvesting

**Only bistability???**

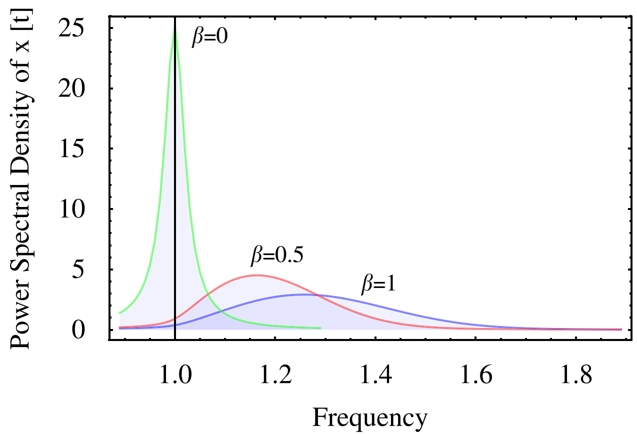
Considering a Duffing oscillator: $\ddot{x} + 2\xi_{eff}\dot{x} + x + \beta x^3 = F(t)$

$\xi_{eff}$ is the effective damping ratio for both electrical and mechanical damping

$\beta > 0$ is a stiffness nonlinearity coefficient




Steady-state frequency response curves under harmonic excitations of a fixed frequency




Power spectral density curves of *x(t)* under White Gaussian excitations of a fixed spectral density.

M.F. Daqaq, Journal of Sound and Vibration 329 (2010) 3621–3631

# Noise energy harvesting

For an electromagnetic duffing-type harvester, (similar to the one shown before):

$$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u + \theta i + \beta u^3 = F(t)$$
$$\theta\dot{u} = iR$$

under a Gaussian White noise :

$$\langle \xi(t)\xi(t_1) \rangle = 2\pi\sigma\delta(t - t_1)$$

*u* represents the position of the mass, $\xi$ is the mechanical damping ratio, $\omega_n$ is the natural frequency, $\beta$ is a cubic nonlinearity coefficient, $\theta$ is an electromechanical coupling coefficient, *R* is the load resistance, and *i* is the current passing through the load.

The expected value of the power (mean power) is: $\langle P \rangle = \dfrac{\theta\sigma}{\xi_{eff}}$ Not dependent on the nonlinear coefficient

For small nonlinearities (small β) and various "colored" noises the mean output power decreases as the nonlinearity increases indicating that the nonlinearity does not improve the output power even when the excitations are Colored.

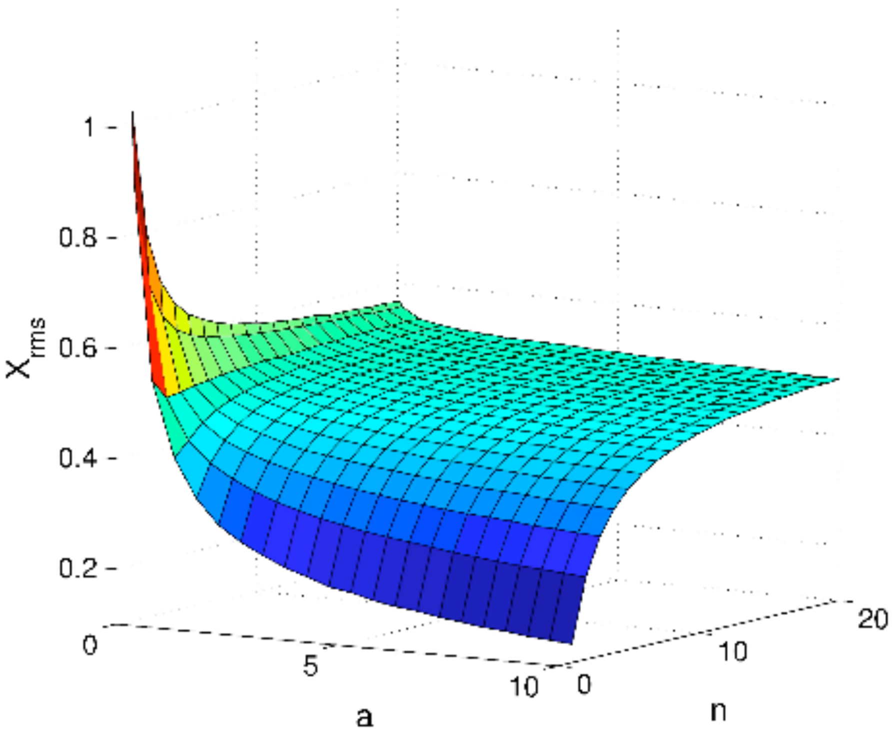
**More details on: M.F. Daqaq, Journal of Sound and Vibration 329 (2010) 3621–3631**

# Noise energy harvesting

**Only bistability???**

A more general monostable potential…

$$U(x) = ax^{2n} \quad \text{with} \quad \begin{array}{l} a > 0 \\ n = 1,2,\ldots \end{array}$$




In an exponentially correlated noise with correlation time τ:

$$\langle \xi(t)\xi(t_1) \rangle = \sigma^2 e^{-\frac{|t-t_1|}{\tau}}$$

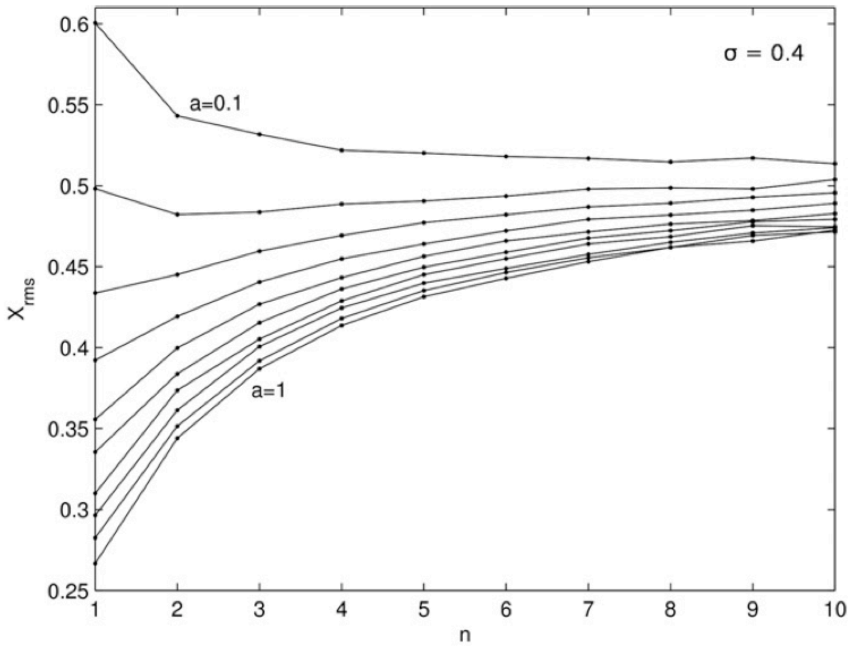
There exists a threshold amplitude $a_{th}$:

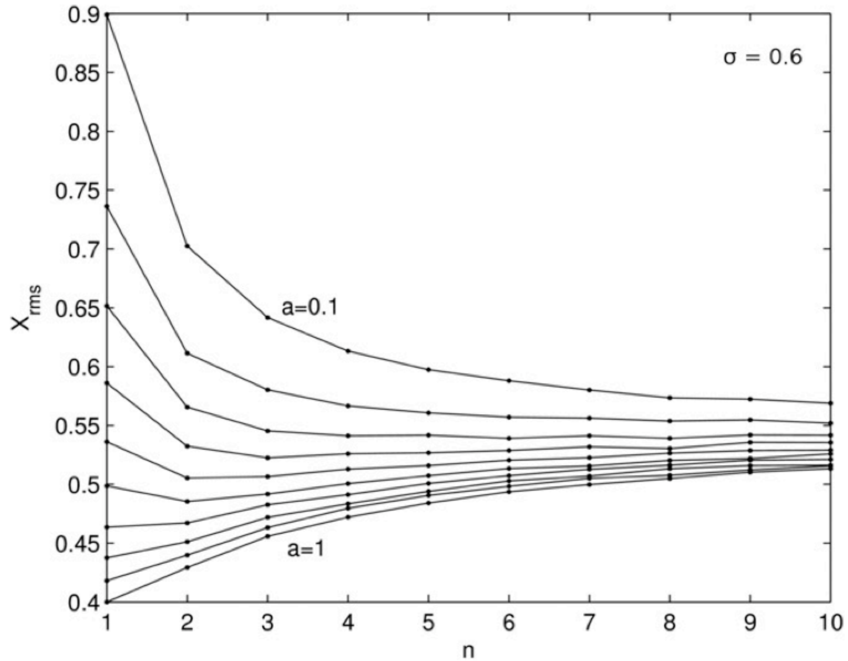
$$a_{th} \approx \frac{D}{4} = \sigma^2 \tau$$

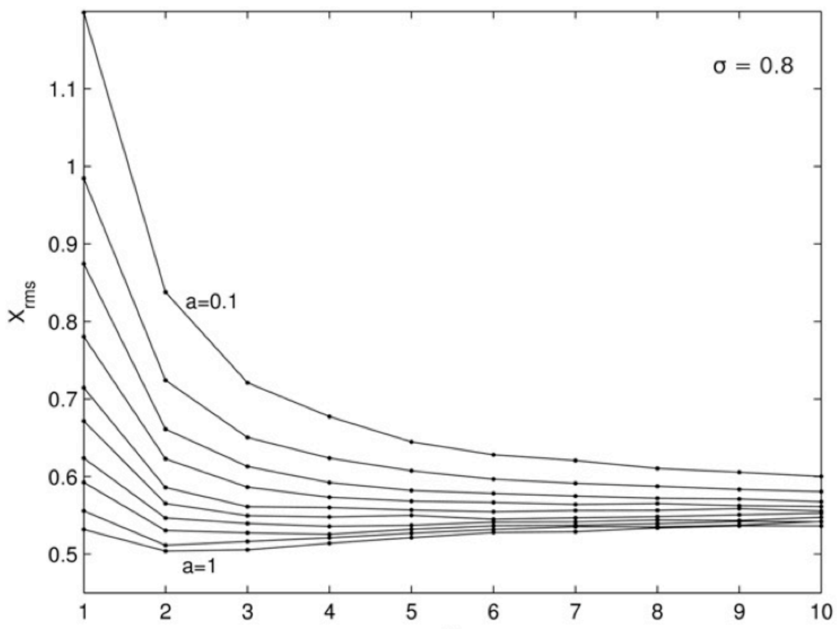
Above which the nonlinear system outperforms the linear one.

L. Gammaitoni, I. Neri, H. Vocca, Appl. Phys. Lett. 94, 164102 (2009)

# Varying the noise amplitude

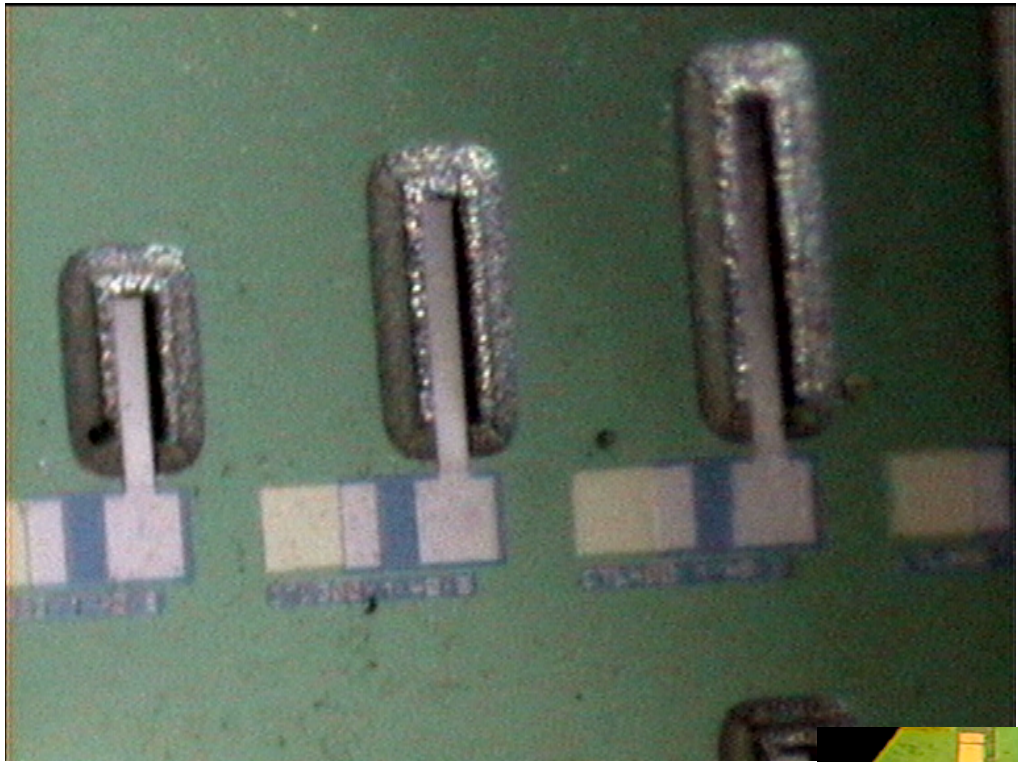


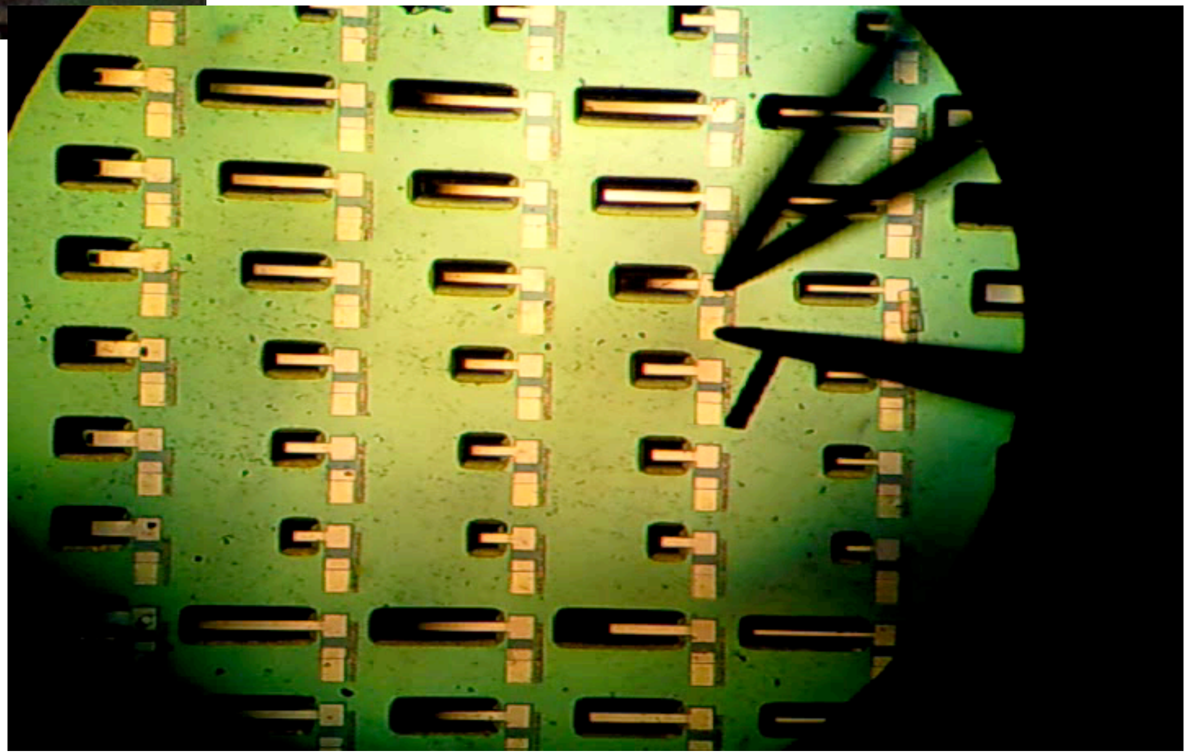




Once σ and a are fixed the choice of a linear (n = 1) or nonlinear potential (n ≥ 2) can be made in order to maximize $x_{rms}$ and consequently the power obtained at the device output.

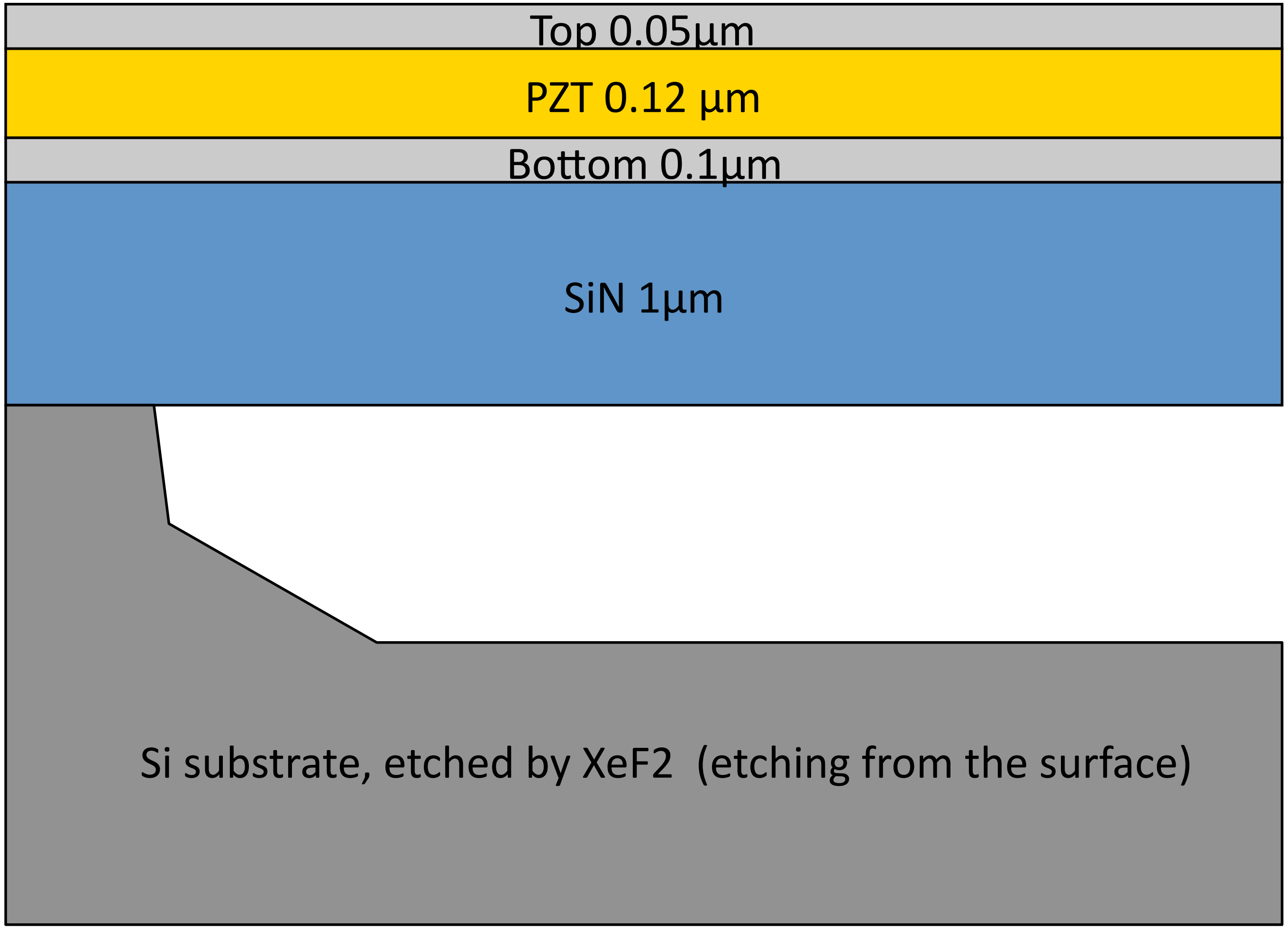
L. Gammaitoni I Neri and H Vocca, Chem. Phys. (2010), doi:10.1016/j.chemphys.2010.08.012



In collaboration with CEA-Leti we are investigating μcantilevers dynamics



# Scheme of the cross section of one cantilever




Typical electrical features

- Max voltage sustainable: 5-7V
- e31 PZT = -5C/m²
- Dielectric constant PZT: 1000

# Conclusions

1) Non resonant (i.e. non-linear) mechanical oscillators can outperform resonant (i.e. linear) ones

2) Non-linear systems are more difficult to treat but more interesting…

3) Bistability is not the only nonlinearity available…

4) The same principles are also valid for capacitive and inductive harvesters

5) A great amount of work has still to be done… good for us!!!

# A macro bistable application has been developed (by wisepower srl)…